STRUCTURE OF NORM CONTINUOUS

QUANTUM MARKOV SEMIGROUP

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Quantum Markov Semigroup (QMS)

$\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$, $\quad \mathcal{T}_t : \mathcal{B}(\mathsf{h}) \to \mathcal{B}(\mathsf{h})$ linear, $\quad$ completely positive,

(semigroup)	$\mathcal{T}_{t+s}(x) = \mathcal{T}_t(\mathcal{T}_s(x))$,	$\mathcal{T}_0(x) = x$
(unital)		$\mathcal{T}_t(\mathbb{1}) = \mathbb{1}$
(normal)	$x_\alpha \uparrow x \quad \Rightarrow$	$\mathcal{T}_t(x_\alpha) \uparrow \mathcal{T}_t(x)$
(norm-continuous)	$\lim_{t\to 0} \sup_{\|x\|\leq 1} \|\mathcal{T}_t(x) - x\| = 0$	
(generator)	$\mathcal{L}(x) = \lim_{t\to 0} t^{-1} (\mathcal{T}_t(x) - x)$	

Problem & Applications

Pr classify QMS with non-trivial decoherence-free subalgebras
A1 structure of invariant states
A2 characterisation of decoherence-free subsystems
A3 characterisation of decoherence-free subspaces

Decoherence-free subalgebra $\mathcal{N}(\mathcal{T})$

Def. $x \in \mathcal{N}(\mathcal{T})$ if and only if

$$\mathcal{T}_t(x^*x) = \mathcal{T}_t(x^*)\mathcal{T}_t(x),\ \mathcal{T}_t(xx^*) = \mathcal{T}_t(x)\mathcal{T}_t(x^*) \quad \forall\, t \geq 0$$

Properties

1. $\mathcal{N}(\mathcal{T})$ is $\mathcal{T}_t$-invariant $\forall t \geq 0$,
2. $\forall\ x \in \mathcal{N}(\mathcal{T})$ and $y \in \mathcal{B}(\mathsf{h})$,
   $\mathcal{T}_t(x^*y) = \mathcal{T}_t(x^*)\mathcal{T}_t(y)$ and $\mathcal{T}_t(y^*x) = \mathcal{T}_t(y^*)\mathcal{T}_t(x)$,
3. $\mathcal{N}(\mathcal{T})$ is a von Neumann subalgebra of $\mathcal{B}(\mathsf{h})$.

Thm (GKSL)

$$\mathcal{L}(x) = G^* x + \Phi(x) + xG, \qquad \text{where } \Phi(x) := \sum_\ell L_\ell^* x L_\ell$$

1. $G, L_\ell \in \mathcal{B}(\mathsf{h}), \quad G^* + \sum_\ell L_\ell^* L_\ell + G = 0,$
2. $\sum_\ell L_\ell^* L_\ell$ strongly convergent,
3. $G = -2^{-1} \sum_\ell L_\ell^* L_\ell - \mathrm{i}H$ with H self-adjoint

Characterisation

$$\begin{aligned}\mathcal{N}(\mathcal{T}) &= \{x \in \mathcal{B}(\mathsf{h}) \mid \mathcal{T}_t(x) = \mathrm{e}^{\mathrm{i}tH} x\, \mathrm{e}^{-\mathrm{i}tH} \quad \forall\, t \geq 0\} \\ &= \{\delta_H^{(n)}(L_\ell), \delta_H^{(n)}(L_\ell^*) \mid n \geq 0, \ell \geq 1\}'\end{aligned}$$

$$\delta_H^{(0)}(L_\ell) = L_\ell,\ \delta_H^{(1)}(L_\ell) = [H, L_\ell],\ \delta_H^{(2)}(L_\ell) = [H,[H,L_\ell]], \ldots$$

Prop. The center $\mathcal{Z}(\mathcal{N}(\mathcal{T})) = \{y \in \mathcal{N}(\mathcal{T}) \mid [y,x] = 0 \, \forall x \in \mathcal{N}(\mathcal{T})\}$ is contained in the set of fixed points

$$\mathcal{F}(\mathcal{T}) = \{x \in \mathcal{B}(\mathsf{h}) \mid \mathcal{T}_t(x) = x \ \forall t \geq 0\}\,.$$

Rem. 1. $\mathcal{F}(\mathcal{T})$ may not be an algebra
2. $\mathcal{F}(\mathcal{T})$ may not contain or be contained in $\mathcal{N}(\mathcal{T})$
3. projections $p \in \mathcal{F}(\mathcal{T})$ belong to $\mathcal{N}(\mathcal{T})$
4. $\mathcal{F}(\mathcal{T}) \subseteq \mathcal{N}(\mathcal{T})$ iff $\mathcal{F}(\mathcal{T})$ is an algebra
5. if $\exists\, \rho$ faithful, normal $\mathcal{T}$-invariant, $\mathcal{F}(\mathcal{T})$ is an algebra

Prop. Projections $p \in \mathcal{F}(\mathcal{T})$ commute with L_ℓ and H. Thus

$$\mathcal{T}_t(xp) = \mathcal{T}_t(xp), \quad \mathcal{T}_t(px) = p\mathcal{T}_t(x), \quad \forall t \geq 0,\ x \in \mathcal{B}(\mathsf{h}).$$

(by $0 = \mathcal{L}(p) = \mathcal{L}(p^2) = p\mathcal{L}(p) + \mathcal{L}(p)p + \sum_\ell |[L_\ell, p]|^2$ ...)

Prop. The center $\mathcal{Z}(\mathcal{N}(\mathcal{T})) = \{y \in \mathcal{N}(\mathcal{T}) \mid [y,x] = 0 \, \forall x \in \mathcal{N}(\mathcal{T})\}$ is contained in the set of fixed points

$$\mathcal{F}(\mathcal{T}) = \{x \in \mathcal{B}(\mathsf{h}) \mid \mathcal{T}_t(x) = x \;\forall t \geq 0\}\,.$$

Structure of $\mathcal{N}(\mathcal{T}) \rightsquigarrow$ structure of $\mathcal{T}$

Assumption (atomicity)

- $\mathcal{N}(\mathcal{T}) = \oplus_i \, p_i \mathcal{N}(\mathcal{T}) \, p_i$
- p_i minimal projections in $\mathcal{Z}(\mathcal{N}(\mathcal{T}))$ s.t. $\sum_i p_i = \mathbb{1}$
- $p_i \mathcal{N}(\mathcal{T}) \, p_i$ type I factor.

The case $\mathcal{N}(\mathcal{T})$ type I factor

Thm. $\mathcal{N}(\mathcal{T})$ type I factor $\Rightarrow$ there exist Hilbert spaces k and m, $U : \mathsf{h} \to \mathsf{k} \otimes \mathsf{m}$ unitary, a GKSL representation of $\mathcal{L}$ by L_ℓ, H s.t.

$$U\mathcal{N}(\mathcal{T})U^* = \mathcal{B}(\mathsf{k}) \otimes \mathbb{1}_{\mathsf{m}}$$

$$UL_\ell U^* = \mathbb{1}_{\mathsf{k}} \otimes M_\ell \qquad UHU^* = K \otimes \mathbb{1}_{\mathsf{m}} + \mathbb{1}_{\mathsf{k}} \otimes M_0$$

with

1. M_ℓ operators on m s.t. $\sum_\ell M_\ell^* M_\ell$ is strongly convergent,
2. K self-adjoint operator on k and M_0 self-adjoint on m.

The case $\mathcal{N}(\mathcal{T})$ type I factor

$\mathcal{N}(\mathcal{T}) \Rightarrow \exists$ Hilbert spaces k and m, $U : \mathsf{h} \to \mathsf{k} \otimes \mathsf{m}$ unitary s.t.

$$U\mathcal{N}(\mathcal{T})U^* = \mathcal{B}(\mathsf{k}) \otimes \mathbb{1}_{\mathsf{m}}$$

If x is in $\mathcal{N}(\mathcal{T})$, then

$$U^* x\, U = x_{\mathsf{k}} \otimes \mathbb{1}_{\mathsf{m}}$$

with x_{k} operator on k.

$UL_\ell U^*, UL_\ell^* U^*$ commute with all operators $x_{\mathsf{k}} \otimes \mathbb{1}_{\mathsf{m}}$ then

$$UL_\ell U^* = \mathbb{1}_{\mathsf{k}} \otimes M_\ell \qquad UL_\ell^* U^* = \mathbb{1}_{\mathsf{k}} \otimes M_\ell^*$$

$\mathcal{N}(\mathcal{T})$ factor: decomposition

$$\widetilde{\mathcal{T}}_t : \mathcal{B}(\mathsf{k}) \otimes \mathcal{B}(\mathsf{m}) \to \mathcal{B}(\mathsf{k}) \otimes \mathcal{B}(\mathsf{m}), \quad \widetilde{\mathcal{T}}_t(a \otimes b) := U \mathcal{T}_t(U^*(a \otimes b)U)U^*$$

$$\widetilde{\mathcal{T}}_t(a \otimes b) = \mathrm{e}^{\mathrm{i}tK} a\, \mathrm{e}^{-\mathrm{i}tK} \otimes \mathcal{T}_t^{\mathsf{m}}(b) \qquad \forall\, a \in \mathcal{B}(\mathsf{k}),\ b \in \mathcal{B}(\mathsf{m}),$$

Generator $\widetilde{\mathcal{L}}$

$$\widetilde{\mathcal{L}}(a \otimes b) = \mathrm{i}\delta_K(a) \otimes b + a \otimes \widetilde{\mathcal{L}}^{\mathsf{m}}(b)$$

where $\delta_K(a) = [K, a]$ and

$$\delta_K \circ \widetilde{\mathcal{L}}^{\mathsf{m}} = \widetilde{\mathcal{L}}^{\mathsf{m}} \circ \delta_K$$

Noiseless / purely dissipative decomposition

$$\widetilde{\mathcal{T}}_t(a \otimes b) \quad = \quad \mathrm{e}^{\mathrm{i}tK} a \, \mathrm{e}^{-\mathrm{i}tK} \quad \otimes \quad \mathcal{T}_t^{\mathsf{m}}(b)$$

noiseless — purely dissipative

Noiseless and purely dissipative factors are "independent"

If $\mathcal{N}(\mathcal{T}) = \oplus_i \, p_i \mathcal{N}(\mathcal{T}) \, p_i$ with p_i minimal projections in $\mathcal{Z}(\mathcal{N}(\mathcal{T}))$ such that $\sum_i p_i = \mathbb{1}$.

$$\mathcal{T}_t(xp_i) = \mathcal{T}_t(x)p_i \quad \mathcal{L}(p_i x) = p_i \mathcal{L}(x) \quad \ldots$$

the same factorisation for each reduced semigroup

$$\mathcal{T}_t^{(i)}(p_i x p_i) := p_i \mathcal{T}_t(x) p_i$$

holds and we find a noiseless / purely dissipative decomposition for the whole $\mathcal{T}$.

$$\widetilde{\mathcal{T}}_t(a \otimes b) \quad = \quad \mathrm{e}^{\mathrm{i}tK} a \, \mathrm{e}^{-\mathrm{i}tK} \quad \otimes \quad \mathcal{T}_t^{\mathrm{m}}(b)$$

noiseless purely dissipative

$$\boxed{\mathcal{N}(\mathcal{T}_t^{\mathrm{m}}) = \mathbb{C}\mathbb{1}_m}$$

Def. A Quantum Markov Semigroup $\mathcal{T}$ is irreducible if $\mathcal{T}_t(p) \geq p$, p projection, $\Rightarrow p = 0$ or $p = \mathbb{1}$.

Structure of normal invariant states

Suppose

1. there exists a faithful normal invariant state ρ
2. (for simplicity) $\mathcal{N}(\mathcal{T}) = \mathcal{B}(\mathsf{k}) \otimes \mathbb{1}_{\mathsf{m}}$ is a factor

Prop. The QMS (purely dissipative) $\mathcal{T}^{\mathsf{m}}$ is irreducible.

Sketchy proof

$$\begin{aligned} 1. \quad &\Rightarrow \quad \mathcal{F}(\mathcal{T}) \subseteq \mathcal{N}(\mathcal{T}) \\ \mathcal{T}_t^{\mathsf{m}}(p) \geq p \quad &\Rightarrow \quad \mathcal{T}_t(\mathbb{1}_{\mathsf{k}} \otimes p) = \mathbb{1}_{\mathsf{k}} \otimes \mathcal{T}_t^{\mathsf{m}}(p) \geq \mathbb{1}_{\mathsf{k}} \otimes p \\ \mathrm{tr}(\rho(\mathcal{T}_t(\mathbb{1}_{\mathsf{k}} \otimes p) - \mathbb{1}_{\mathsf{k}} \otimes p)) = 0 \quad &\Rightarrow \quad \mathcal{T}_t(\mathbb{1}_{\mathsf{k}} \otimes p) = \mathbb{1}_{\mathsf{k}} \otimes p \in \mathcal{F}(\mathcal{T}) \\ \mathbb{1}_{\mathsf{k}} \otimes p \in \mathcal{N}(\mathcal{T}) \quad &\Rightarrow \quad \mathbb{1}_{\mathsf{k}} \otimes p = q \otimes \mathbb{1}_{\mathsf{m}} \Rightarrow \; p = \mathbb{1}_{\mathsf{m}} \end{aligned}$$

Cor. $\mathcal{T}^{\mathsf{m}}$ has a unique faithful normal invariant state τ.

Structure of invariant states 2

$$\mathcal{T}_t(a \otimes b) \;=\; \mathrm{e}^{\mathrm{i}tK} a\, \mathrm{e}^{-\mathrm{i}tK} \otimes \mathcal{T}^{\mathsf{m}}_t(b)$$

Thm. Suppose that there exists a faithful normal invariant state ρ. Any normal $\mathcal{T}$-invariant normal state η is

$$\eta = \sigma \otimes \tau$$

where τ is the unique (faithful) $\mathcal{T}^{\mathsf{m}}$-invariant normal state and σ is a normal state on $\mathcal{B}(\mathsf{h})$ with density belonging to $\{K\}'$.

Proof.

Step 1. Partial traces ρ_{k}, ρ_{m} of ρ on each factor are invariant states of the factor semigroups.

Step 2. $\mathrm{e}^{-\mathrm{i}tK}\rho_{\mathsf{k}}\mathrm{e}^{\mathrm{i}tK} = \rho_{\mathsf{k}}\ \forall t \geq 0 \Rightarrow K$ pure point spectrum

Step 3. $\mathcal{F}(\mathcal{T}^{\mathsf{m}}) = \mathcal{N}(\mathcal{T}^{\mathsf{m}})(= \mathbb{C}\mathbb{1}_{\mathsf{m}})$ implies

$$\lim_{t\to\infty} \mathcal{T}^{\mathsf{m}}_{*t}(\omega) = \mathrm{tr}_{\mathsf{m}}(\omega)\,\tau$$

Step 4. $Kf_j = \alpha_j f_j$

$$\eta = \sum_{j,k} |f_j\rangle\langle f_k| \otimes \eta_{jk}$$

$$\eta = \mathcal{T}_{*t}(\eta) = \sum_{j,k} \mathrm{e}^{\mathrm{i}(\alpha_j - \alpha_k)t} |f_j\rangle\langle f_k| \otimes \mathcal{T}^{\mathrm{m}}_{*t}(\eta_{jk})$$

Step 5. $\eta = t^{-1}\int_0^t \mathcal{T}_{*s}(\eta)ds$

$$\approx \sum_{j,k\, \alpha_j \neq \alpha_k} \frac{\mathrm{e}^{\mathrm{i}(\alpha_j-\alpha_k)t} - 1}{\mathrm{i}(\alpha_j - \alpha_k)t} |f_j\rangle\langle f_k| \otimes \mathcal{T}^{\mathrm{m}}_{*t}(\eta_{jk}) \; (t \to \infty) \to 0$$

$$+ \sum_{j,k\, \alpha_j = \alpha_k} |f_j\rangle\langle f_k| \otimes \mathcal{T}^{\mathrm{m}}_{*t}(\eta_{jk}) \to \sum_{j,k\, \alpha_j = \alpha_k} \mathrm{tr}_{\mathrm{m}}(\eta_{jk}) |f_j\rangle\langle f_k| \otimes \tau$$

$\mathcal{N}(\mathcal{T})$ not a factor

$$\mathcal{N}(\mathcal{T}) = \oplus_i \, p_i \, \mathcal{N}(\mathcal{T}) \, p_i$$

p_i minimal projections in $\mathcal{Z}(\mathcal{N}(\mathcal{T}))$ such that $\sum_i p_i = \mathbb{1}$.

$$\mathsf{h} = \bigoplus\nolimits_i \mathsf{k}_i \otimes \mathsf{m}_i, \qquad \mathcal{L} = \sum_i \Big(\mathsf{i}\, [\, K_i, \cdot\,] \otimes \mathsf{Id}_{\mathcal{B}(\mathsf{m}_i)} + \mathsf{Id}_{\mathcal{B}(\mathsf{k}_i)} \otimes \mathcal{L}^{\mathsf{m_i}} \Big)$$

Invariant states

$$\sum_i \lambda_i \;\sigma_i \otimes \tau_i, \qquad \lambda_i \geq 0, \; \sum_i \lambda_i = 1$$

Decoherence free subsystems

Intuitive idea: there is a certain decomposition

$$\mathsf{h} = (\mathsf{h}_{\mathsf{df}} \otimes \mathsf{h}_F) \oplus \mathsf{h}_R$$

and the evolution of states supported on h_{df} is unitary.

Decoherence free subspace when $\mathsf{h}_F \simeq \mathbb{C}$.

Def. h_{df} supports a decoherence free (df) subsystem for some $\mathcal{T}$ iff for all initial state $\rho = \rho_{\mathsf{df}} \otimes \rho_F$, its evolution is given by

$$\mathcal{T}_{*t}(\rho) = \left(\begin{array}{c|c} U_t \rho_{\mathsf{df}} U_t^* \otimes \mathcal{T}^F_{*t}(\rho_F) & 0 \\ \hline 0 & 0 \end{array} \right)$$

where U_t is a unitary operator on h_{df} and $\mathcal{T}^F$ QMS on $\mathcal{B}(\mathsf{h}_F)$.

Decoherence free subsystems

$$\mathcal{N}(\mathcal{T}) = \oplus_i \, p_i \, \mathcal{N}(\mathcal{T}) \, p_i$$

p_i minimal projections in $\mathcal{Z}(\mathcal{N}(\mathcal{T}))$ such that $\sum_i p_i = \mathbb{1}$.

$$\mathsf{h} = \bigoplus_i \mathsf{k}_i \otimes \mathsf{m}_i, \qquad \mathcal{L} = \sum_i \Big(\mathsf{i} \, [\, K_i, \, \cdot \,] \otimes \mathrm{Id}_{\mathcal{B}(\mathsf{m}_i)} + \mathrm{Id}_{\mathcal{B}(\mathsf{k}_i)} \otimes \mathcal{L}^{\mathsf{m_i}} \Big)$$

$$\mathsf{h} = (\mathsf{h}_{\mathsf{df}} \otimes \mathsf{h}_F) \oplus \mathsf{h}_R, \qquad \mathcal{T}_{*t}(\rho) = \left(\begin{array}{c|c} U_t \rho_{\mathsf{df}} U_t^* \otimes \mathcal{T}_{*t}^F(\rho_F) & 0 \\ \hline 0 & 0 \end{array} \right)$$

$\mathbf{k}_i$ are all df subsystems

df subspaces

Def. A subspace h_{df} of h is *decoherence-free* if there exists a self-adjoint R on h with $R\mathsf{h}_{\mathsf{df}} \subseteq \mathsf{h}_{\mathsf{df}}$ s.t. for all normal state η with support in h_{df} we have

$$\mathcal{T}_{*t}(\eta) = \mathrm{e}^{-\mathrm{i}tR}\eta\,\mathrm{e}^{\mathrm{i}tR}, \qquad \forall t \geq 0.$$

Prop. Suppose that: $\mathcal{N}(\mathcal{T})$ is an atomic algebra, ρ faithful normal $\mathcal{T}$-invariant state, K_i has pure point spectrum for all $i \in I$, $\mathsf{h} \simeq \oplus_{i\in I}(\mathsf{k}_i \otimes \mathsf{m}_i)$. If $\mathsf{h}_{\mathsf{df}} \subseteq \mathsf{h}$ is a maximal **df** subspace with Hamiltonian R having a discrete spectrum, then

$$\mathsf{h}_{\mathsf{df}} \simeq \oplus_{i\in J}(\mathsf{k}_i \otimes \mathsf{m}_i) \simeq \oplus_{i\in J}\ \mathsf{k}_i$$

for $J = \{i \in I \mid \dim(\mathsf{m}_i) = 1\}$ and $R = \oplus_{i\in J} K_i$

df subspaces

Indeed, for a factor $p_i\mathcal{N}(\mathcal{T})p_i$,

$$\mathcal{T}_{*t}^{(i)}(\omega_i \otimes \tau_i) = \mathsf{e}^{-\mathsf{i}tK_i}\omega_i\mathsf{e}^{\mathsf{i}tK_i} \otimes \mathcal{T}_{*t}^{\mathsf{m}_i}(\tau_i)$$

$\mathcal{T}_{*t}^{\mathsf{m}_i}(\tau_i) \neq \mathsf{e}^{-\mathsf{i}tS_i}\tau_i\mathsf{e}^{\mathsf{i}tS_i}$ for some self-adjoint S_i unless $\dim(\mathsf{m}_i) = 1$

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Thank you!