On Totality of Certain Sets of Exponential Vectors

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Fock space and exponential vectors

The symmetric Fock space over a Hilbert space H is

$$\Gamma(H) := \bigoplus_{n\in\mathbb{N}_0} H^{\otimes_s n},$$

where $H^{\otimes_s n} = \overline{\operatorname{span}}\{x^{\otimes n} \colon x \in H\} \subset H^{\otimes n}$ $(n \geq 1)$ and $H^{\otimes_s 0} = \mathbb{C}\Omega$.

The exponential vector to $x \in H$ is $\mathbb{e}(x) := \sum_{n\in\mathbb{N}_0} \frac{x^{\otimes n}}{\sqrt{n!}}$.

<u>Easy:</u> The exponential vectors form a total subset of $\Gamma(H)$.

Indeed, $\left(\frac{d}{d\lambda}\right)^n\Big|_{\lambda=0} \mathbb{e}(\lambda x) = \sqrt{n!}x^{\otimes n}$.

In applications it is useful to find subsets $S \subset H$ such that $\mathbb{e}(S)$ is still total.
For instance, by continuity of $x \mapsto \mathbb{e}(x)$, any dense subspace of H is enough. (Or von Neumann for $H = \mathbb{C}$.)
We can do much better:

Parthasarathy and Sunder 1998 (-12)

Denote by $\mathfrak{I}_I \subset L^2(\mathbb{R}_+)$ the set of (meas., L^2) indicator functions.

Theorem ([PS98])

The set $\mathbb{e}(\mathfrak{I}_I)$ *is total in* $\Gamma(L^2(\mathbb{R}_+))$.

Note:

- If $\mathbb{I} \in \mathfrak{I}_I$, then $\lambda\mathbb{I} \in \mathfrak{I}_I$ iff $\mathbb{I} = 0$ or $\lambda = 0$.
- By continuity, it is enough to take only step functions.

Proof by [PS98] (≤ 1986).

By reduction to the martingale convergence theorem and some not so easy estimates. □

Proof by Bhat [Bha01] (≤ 1998).

Applying his results on minimality of Evans-Hudson dilation obtained via quantum stochastic calculus to a cleverly chosen Markov semigroup on M_2. □

My proof [Ske00] (1999)

(Inspired very much by Arveson [Arv89, Proposition 6.3].)
Essentially:

- For $0 \leq \varkappa \leq 1$, we get $\mathbb{I}_{\bigcup_{k=1}^{n}[\frac{k-1}{n},\frac{k-1}{n}+\frac{\varkappa}{n}]} \to \varkappa \mathbb{I}_{[0,1]}$, weakly.
- So, $\mathbb{e}\Big(\mathbb{I}_{\bigcup_{k=1}^{n}[\frac{k-1}{n},\frac{k-1}{n}+\frac{\varkappa}{n}]}\Big) \to \mathbb{e}(\varkappa \mathbb{I}_{[0,1]})$, weakly.
  Indeed, we have $\langle \mathbb{e}(x), \mathbb{e}(y) \rangle = e^{\langle x,y \rangle}$. So:
  - $\Big\|\mathbb{e}\Big(\mathbb{I}_{\bigcup_{k=1}^{n}[\frac{k-1}{n},\frac{k-1}{n}+\frac{\varkappa}{n}]}\Big)\Big\| \leq \sqrt{e} \quad \rightsquigarrow$ check only with $\mathbb{e}(x)$.
  - Weak convergence in L^2 lifts to weak convergence in Γ.
- For subspaces of Hilbert space, weak closure=norm closure.
- Appropriate generalization of approximation of $\mathbb{e}(\varkappa \mathbb{I}_{[0,1]})$ gives approximation of all step functions with values in $[0, 1]$.
  ($\rightsquigarrow$ ready to do redo proof by differentiation.) □

The last step is easy but cumbersome to be written down.
Better: Product systems!

Corollary (MS [Ske00]). $0 \in S$ a total subset of $K \rightsquigarrow$ exponential vectors to S–valued stepfunctions are total in $\Gamma(L^2(\mathbb{R}_+, K))$.

Fock space as product system

Recall that $\Gamma(H) \otimes \Gamma(G) \cong \Gamma(H \oplus G)$ via $\mathbb{e}(x) \otimes \mathbb{e}(y) \mapsto \mathbb{e}(x+y)$. Then

$$\Gamma_t \ := \ \Gamma(L^2([0,t],K)) \quad (\subset \ \Gamma(L^2(\mathbb{R}_+,K)) \ ).$$

give product system $u_{s,t} \colon \Gamma_s \otimes \Gamma_t \to \Gamma_{s+t}$ with (associative) product $X_s Y_t := u_{s,t}(X_s \otimes Y_t)$

$$\mathbb{e}(x_s)\mathbb{e}(y_t) \ = \ \mathbb{e}(\mathcal{S}_t x_s + y_t).$$

With this,

$$\mathbb{e}\Big(\mathbb{I}_{\bigcup_{k=1}^{n}[\frac{k-1}{n},\frac{k-1}{n}+\frac{\varkappa}{n}]}\Big) \ = \ \Big(\mathbb{e}(0 \cdot \mathbb{I}_{[0,\frac{1-\varkappa}{n}]})\mathbb{e}(1 \cdot \mathbb{I}_{[0,\frac{\varkappa}{n}]})\Big)^n \xrightarrow[\text{weakly}]{} \mathbb{e}\Big((\varkappa \cdot 1 + (1-\varkappa) \cdot 0)\mathbb{I}_{[0,1]}\Big)$$

Much better:

Theorem (Very special case of Liebscher-MS [LS08])

$$\Big((\varkappa_2 \mathbb{e}(k_2 \cdot \mathbb{I}_{[0,\frac{1}{n}]}) + \varkappa_1 \mathbb{e}(k_1 \cdot \mathbb{I}_{[0,\frac{1}{n}]})\Big)^n \xrightarrow[\text{norm}]{} \mathbb{e}\Big((\varkappa_1 k_1 + \varkappa_2 k_2)\mathbb{I}_{[0,1]}\Big)$$

for all $\varkappa_i \in \mathbb{C}$ with $\varkappa_1 + \varkappa_2 = 1$.

Applications

Many application around quantum stochastic calculus.
Our applications:

- The realization of a QLP obtained by Schürmann [Sch93] by resolving a QSDE (starting from a minimal Schürmann triple for the generator), is cyclic.
  Idea: Take some vectors from the process, apply some Trotter-like procedure and get a total set of exponential vectors.
  (Weakly. Unpublished from 2002; see Franz [Fra06] (2003).)
- Reverse this procedure to get a realization of an arbitrary QLP out of Trotter products of exponential vectors.
  Idea: A new version of product system valued L^2–QLPs.
  (Strongly. MS 2014; yet unpublished.)
  Can be viewed as a special case of:
- Schürmann-MS-Volkwardt [SSV10]: Transformations of QLP.

I would have liked to explain in detail ....

Thank you!

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