Quantum automorphism groups of finite quantum groups

Algebraic and Analytic Aspects
of Quantum Lévy Processes

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Plan of talk

1. Finite quantum groups
2. Quantum automorphism group of a finite q.g.
3. A commutativity result
4. Commutativity of $\mathrm{C}(\mathbb{G})$

Finite quantum groups

- Let (A,Δ) be a finite quantum group i.e.
  - A is a finite dimensional C^*-algebra,
  - (A,Δ) is a Hopf $*$-algebra.
- Let $\boldsymbol{h}$ be the Haar measure of (A,Δ):

$$(\boldsymbol{h}\otimes\mathrm{id})\Delta(a)=\boldsymbol{h}(a)\mathbb{1}=(\mathrm{id}\otimes\boldsymbol{h})\Delta(a),\qquad a\in\mathsf{A}.$$

- Let $\mathcal{H}$ be the GNS-Hilbert space for $(\mathsf{A},\boldsymbol{h})$, so that $\mathsf{A}\subset\mathrm{B}(\mathcal{H})$.
- As $\mathcal{H}\cong\mathsf{A}$ the map

$$\mathsf{A}\otimes\mathsf{A}\ni a\otimes b\longmapsto\Delta(a)(\mathbb{1}\otimes b)\in\mathsf{A}\otimes\mathsf{A}$$

  can be transported to an operator $W\in\mathrm{B}(\mathcal{H}\otimes\mathcal{H})$.
- W is unitary and

$$W_{23}W_{12}W_{23}^*=W_{12}W_{13}$$

  on $\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}$.

Finite quantum groups

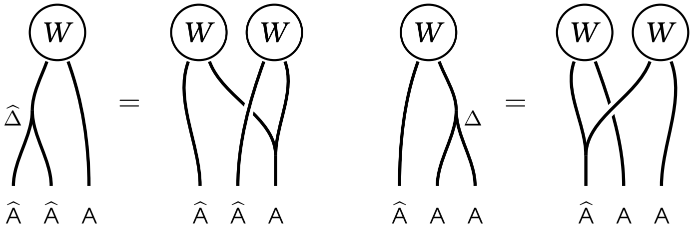
- We have
$$\mathsf{A}=\bigl\{(\omega\otimes\mathrm{id})(W)\bigm|\omega\in\mathrm{B}(\mathcal{H})^*\bigr\},$$
so $W\in\mathrm{B}(\mathcal{H})\otimes\mathsf{A}$.
- For $a\in\mathsf{A}$ the operator of multiplication by $\Delta(a)$ on $\mathsf{A}\otimes\mathsf{A}\cong\mathcal{H}\otimes\mathcal{H}$ is
$$W(a\otimes\mathbb{1})W^*.$$
- It follows that $(\mathrm{id}\otimes\Delta)(W)=W_{12}W_{13}$.
- Define
$$\widehat{\mathsf{A}}=\bigl\{(\mathrm{id}\otimes\omega)(W)\bigm|\omega\in\mathrm{B}(\mathcal{H})^*\bigr\}.$$
Then $W\in\widehat{\mathsf{A}}\otimes\mathsf{A}$.
- The map $\Gamma\colon\mathsf{A}^*\ni\varphi\mapsto(\mathrm{id}\otimes\varphi)(W)\in\widehat{\mathsf{A}}$ is an isomorphism of vector spaces.
- A^* carries a Hopf $*$-algebra structure, and Γ is a $*$-algebra isomorphism.

FINITE QUANTUM GROUPS

- Transporting the comultiplication from A^* to $\widehat{\mathsf{A}}$ (via Γ) we obtain $\widehat{\Delta}\colon \widehat{\mathsf{A}} \to \widehat{\mathsf{A}} \otimes \widehat{\mathsf{A}}$ and

$$(\widehat{\Delta} \otimes \mathrm{id})(W) = W_{13}W_{23}.$$

- Pictures:



Quantum automorphisms of quantum groups

- Let (A, Δ) be a finite quantum group, let B be a C*-algebra. A $*$-homomorphism $\alpha\colon \mathsf{A} \to \mathsf{A} \otimes \mathsf{B}$ represents a **quantum family of invertible maps** if

$$\alpha(\mathsf{A})(\mathbb{1} \otimes \mathsf{B}) = \mathsf{A} \otimes \mathsf{B}. \qquad \text{(Podleś condition)}$$

- Given $\alpha\colon \mathsf{A} \to \mathsf{A} \otimes \mathsf{B}$ define a linear map $\widehat{\alpha}\colon \widehat{\mathsf{A}} \to \widehat{\mathsf{A}} \otimes \mathsf{B}$ by

$$\widehat{\alpha} = (\mathcal{F} \otimes \mathrm{id}) \circ \alpha \circ \mathcal{F}^{-1},$$

where

$$\mathcal{F}\colon \mathsf{A} \ni a \longmapsto \big(\mathrm{id} \otimes \boldsymbol{h}(\cdot a)\big)(W) \in \widehat{\mathsf{A}}$$

is the **Fourier transform**.

Quantum automorphisms of quantum groups

Theorem

Let $\alpha\colon \mathsf{A} \to \mathsf{A} \otimes \mathsf{B}$ represent a quantum family of invertible maps. Then the following are equivalent:

1. *α preserves the convolution product on A, the convolution adjoint and the Haar element;*
2. *$\widehat{\alpha}$ represents a quantum family of invertible maps.*

Moreover in this case $\widehat{\widehat{\alpha}} = \alpha$.

- Convolution: $a \star b = (\boldsymbol{h} \otimes \mathrm{id})\Bigl(\bigl((S \otimes \mathrm{id})\Delta(b)\bigr)(a \otimes \mathbb{1})\Bigr)$.
- Convolution adjoint: $a^{\bullet} = S(a)^*$.
- Haar element: there exists $\eta \in \mathsf{A}$ such that for all $a \in \mathsf{A}$

$$a\eta = \eta a = \varepsilon(a)\eta.$$

Quantum automorphisms of quantum groups

Theorem

Let $\alpha\colon \mathsf{A} \to \mathsf{A} \otimes \mathsf{B}$ *represent a quantum family of invertible maps. Then the following are equivalent:*

1. α *preserves the convolution product on* A*, the convolution adjoint and the Haar element;*
2. $\widehat{\alpha}$ *represents a quantum family of invertible maps.*

Moreover in this case $\widehat{\widehat{\alpha}} = \alpha$*.*

Definition

A q.f.i.m. $\alpha\colon \mathsf{A} \to \mathsf{A} \otimes \mathsf{B}$ is a **quantum family of automorphisms** of (A, Δ) when the conditions of the theorem are satisfied.

QUANTUM AUTOMORPHISMS OF QUANTUM GROUPS

THEOREM

1. *There exists a q.f.a.* $\boldsymbol{\alpha}\colon \mathsf{A} \to \mathsf{A} \otimes \mathsf{S}$ *such that for any q.f.a.* $\beta\colon \mathsf{A} \to \mathsf{A} \otimes \mathsf{B}$ *there exists a unique* $\Lambda\colon \mathsf{S} \to \mathsf{B}$ *such that*

$$\beta = (\mathrm{id} \otimes \Lambda)\circ\boldsymbol{\alpha}.$$

2. S *carries a structure of the* C^**-algebra of functions on a compact quantum group* $\mathbb{G}$ *and* $\boldsymbol{\alpha}$ *is and action of* $\mathbb{G}$ *on* A *(*$\mathbb{G}$ *is the* ***quantum automorphism group*** *of* (A, Δ)*).*
3. *The Haar measure* $\boldsymbol{h}$ *is invariant for* $\boldsymbol{\alpha}$*:*

$$(\boldsymbol{h} \otimes \mathrm{id})\boldsymbol{\alpha}(a) = \boldsymbol{h}(a)\mathbb{1}, \qquad a \in \mathsf{A}.$$

4. *The quantum automorphism group of* $(\widehat{\mathsf{A}}, \widehat{\Delta})$ *is canonically isomorphic to that of* (A, Δ)*.*

Quantum automorphisms of quantum groups

- Let $\mathbb{G}$ be the quantum automorphism group of (A,Δ) and let

$$\boldsymbol{\alpha}\colon \mathsf{A} \longrightarrow \mathsf{A} \otimes \mathrm{C}(\mathbb{G})$$

  be its action on A.
- $\mathbb{G}$ is of Kac type — this is related to invariance of $\boldsymbol{h}$ under $\boldsymbol{\alpha}$.
- The C^*-algebra $\mathrm{C}(\mathbb{G})$ is generated by

$$\bigl\{(\omega \otimes \mathrm{id})\boldsymbol{\alpha}(a)\bigm| a \in \mathsf{A},\ \omega \in \mathsf{A}^*\bigr\}.$$

- The Gelfand spectrum of $\mathrm{C}(\mathbb{G})$ (the **classical points** of $\mathbb{G}$) is naturally identified with the set of Hopf $*$-automorphisms of the Hopf $*$-algebra (A,Δ).

A commutativity result

Theorem

Let C *be a* C^**-algebra and let*

$$\beta\colon \mathsf{A} \longrightarrow \mathsf{C} \otimes \mathsf{A} \quad \textit{and} \quad \gamma\colon \widehat{\mathsf{A}} \longrightarrow \widehat{\mathsf{A}} \otimes \mathsf{C}$$

be $*$*-homomorphisms such that*

$$(\mathrm{id} \otimes \beta)(W) = (\gamma \otimes \mathrm{id})(W).$$

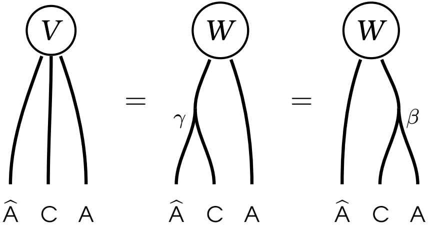
Then the algebra generated by

$$\bigl\{(\mathrm{id} \otimes \omega)\beta(a) \bigm| a \in \mathsf{A},\, \omega \in \mathsf{A}^*\bigr\} \subset \mathsf{C}$$

is commutative.

PROOF:

Define $V = (\mathrm{id} \otimes \beta)(W) \in \widehat{\mathsf{A}} \otimes \mathsf{C} \otimes \mathsf{A}$ (then also $V = (\gamma \otimes \mathrm{id})(W)$).
In pictures:



We have

$$(\widehat{\Delta} \otimes \mathrm{id} \otimes \mathrm{id})(V) = V_{134} V_{234}$$

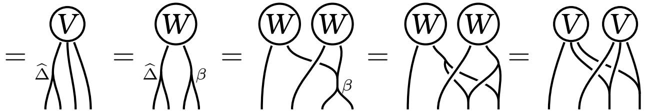
and

$$(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(V) = V_{123} V_{124}.$$

PROOF:

Indeed:

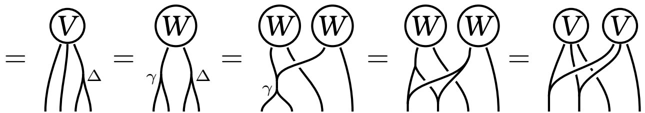
$(\widehat{\Delta} \otimes \mathrm{id} \otimes \mathrm{id})(V) =$



$= V_{134} V_{234}.$

Similarly:

$(\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(V) =$

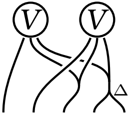


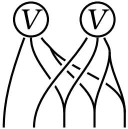
$= V_{123} V_{124}.$

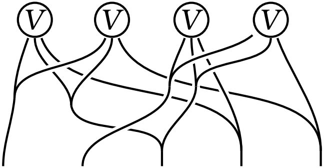
This gives us two ways of computing $(\widehat{\Delta} \otimes \mathrm{id} \otimes \Delta)(V)$.

Proof:

On one hand
$(\widehat{\Delta}\otimes\mathrm{id}\otimes\Delta)(V)=(\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id}\otimes\Delta)(V_{134}V_{234})=$

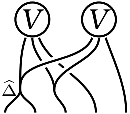
$=$ 

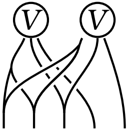
$=$ 

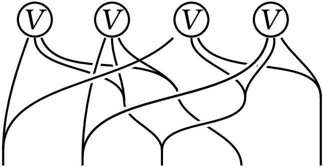
$=$ 

$=V_{134}V_{135}V_{234}V_{235}.$

And on the other
$(\widehat{\Delta}\otimes\mathrm{id}\otimes\Delta)(V)=(\widehat{\Delta}\otimes\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{id})(V_{123}V_{124})=$

$=$ 

$=$ 

$=$ 

$=V_{134}V_{234}V_{135}V_{235}.$

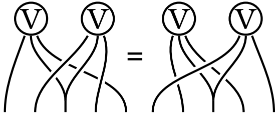
PROOF:

It follows that

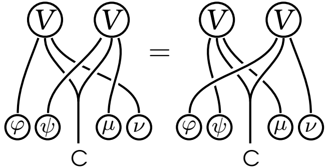
$$V_{134}V_{135}V_{234}V_{235} = V_{134}V_{234}V_{135}V_{235}$$

and, since V is unitary, we obtain

$$V_{135}V_{234} = V_{234}V_{135} \qquad \text{i.e.}$$

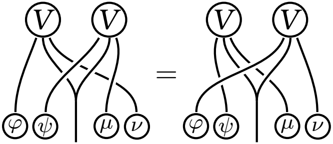


Apply $(\varphi \otimes \psi \otimes \mathrm{id} \otimes \mu \otimes \nu)$ to both sides:

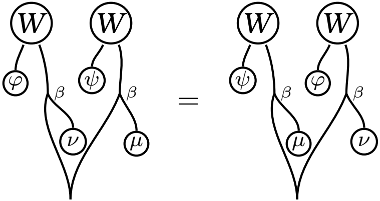


PROOF:

From



(recalling that $V = (\mathrm{id} \otimes \beta)(W)$) we get



Which shows that
$(\mathrm{id} \otimes \nu)\beta(a)(\mathrm{id} \otimes \mu)\beta(b) = (\mathrm{id} \otimes \mu)\beta(b)(\mathrm{id} \otimes \nu)\beta(a)$ for all $a, b \in \mathsf{A}$.

□

$\mathbb{G}$ is classical

- Let $\alpha\colon \mathsf{A} \to \mathsf{A} \otimes \mathrm{C}(\mathbb{G})$ be the action of the quantum automorphism group of (A, Δ).
- Since $\mathbb{G}$ is of Kac type the map

$$\gamma = \sigma \circ (S \otimes S_{\mathbb{G}}) \circ \alpha \circ S\colon \mathsf{A} \longrightarrow \mathrm{C}(\mathbb{G}) \otimes \mathsf{A}$$

is a $*$-homomorphism.

Theorem

We have

$$(\mathrm{id} \otimes \gamma)(W) = (\widehat{\alpha} \otimes \mathrm{id})(W).$$

Corollary

The algebra $\mathrm{C}(\mathbb{G})$ is commutative. In particular, $\mathbb{G}$ is the classical group of Hopf $*$-algebra automorphisms of (A, Δ).

Graduate school on topological quantum groups

June 28 – July 11, 2015, Będlewo, Poland

Speakers:

- Teodor Banica
- Sergey Neshveyev
- Kenny De Commer
- Roland Speicher
- Martijn Caspers
- Reiji Tomatsu
- Michael Brannan
- Zhong-Jin Ruan

http://bcc.impan.pl/15TQG/