Unitary quantum stochastic double product integrals as implementors of Bogolubov transformations.

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Abstract

Volterra originally conceived the notion of product integral as a multiplicative analog of the Riemann or Lebesgue integral, that is as a limit of discrete products rather than discrete sums. Such product integrals provided useful descriptions of solutions of systems of differential equations. But they were not originally *defined* as such solutions.

Here we consider a triple generalization, *quantum, stochastic, double* product integrals, of Volterra's concept. Thus the domain of integration becomes a planar area such as a rectangle or triangle, the calculus used becomes stochastic so that Itô terms may appear, and finally the theory becomes noncommutative or quantum in character. Following Volterra's philosophy we define them nonrigorously as limits of discrete double products which are well-defined because of, and are manipulated using, quantum commutation relations.

By combining some already known examples using Trotter product type constructions, we express the general unitary-valued rectangular quantum stochastic double product integral as the unitary implementor of an explicitly determined Bogolubov transformation. Rigour is achieved *a posteriori.*

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- The constructions are mathematically intuitive. Rigor is achieved *a posteriori*, by showing that qsde's are satisfied.

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- An immediate difficulty arises when the elements $x_{j,k}$ do not

  commute; *how to order these products?* Everyone agrees that $\Pi_{1\leq j\leq N}x_j$ means $x_1x_2...x_N$, *but* is

$$\begin{aligned}\Pi_{(j,k)\in\mathbb{N}_m\times\mathbb{N}_n}x_{j,k} &= \Pi_{1\leq j\leq m}\left\{\Pi_{1\leq k\leq n}x_{j,k}\right\}?\ \Pi_{1\leq k\leq n}\left\{\Pi_{1\leq j\leq m}x_{j,k}\right\}?\\ \Pi_{1\leq j<k\leq N}x_{j,k} &= \Pi_{1\leq j<N}\left\{\Pi_{j<k\leq N}x_{j,k}\right\}?\ \Pi_{1<k\leq N}\left\{\Pi_{1\leq j<k}x_{j,k}\right\}?\end{aligned}$$

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- All ambiguities are resolved if the $x_{j,k}$ are *weakly commuting*, ie $[x_{j,k}, x_{j',k'}] = 0$ if *both* $j \neq j'$ *and* $k \neq k'$. Then, for example,

$$\begin{aligned} & \Pi_{1\leq j\leq 2}\left\{\Pi_{1\leq k\leq 2} x_{j,k}\right\} = x_{1,1}x_{1,2}x_{2,1}x_{2,2} = x_{1,1}x_{2,1}x_{1,2}x_{2,2}\\ = & \Pi_{1\leq k\leq 2}\left\{\Pi_{1\leq j\leq 2} x_{j,k}\right\},\\ & \Pi_{1\leq j<4}\left\{\Pi_{j<k\leq 4} x_{j,k}\right\} = x_{1,2}x_{1,3}x_{1,4}x_{2,3}x_{2,4}x_{3,4}\\ = & x_{1,2}x_{1,3}x_{2,3}x_{1,4}x_{2,4}x_{3,4} = \Pi_{1<k\leq 4}\left\{\Pi_{1\leq j<k} x_{j,k}\right\}.\end{aligned}$$

- We construct continuous rectangular and triangular products

$$\prod_{[a,b[\times[c,d[} (1+dr)\,,\ \prod_{\Delta_{[a,b[}} (1+dr)$$

where $\Delta_{[a,b[} = \left\{(s,t)\in\mathbb{R}^2 : a \le s\ < t < b\right\}$ and $dr \in \mathcal{I}^2$ is a second rank tensor over the Itô algebra $\mathcal{I} = \mathbb{C}\left\langle dA^{\dagger}, dA, dT\right\rangle = \mathbb{C}\left\langle dP, dQ, dT\right\rangle$. Here $A^{\dagger}$ and A are the standard *creation* and *annihilation* processes of quantum stochastic calculus with self-adjoint real and imaginary parts Q and P, the *position* and *momentum* Brownian motions, satisfying

$$[P(s), Q(t)] = -2iT(s\wedge t)\,; \tag{1}$$

where T is the time process; $T(t) = tI$.

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$$[P(s), Q(t)] = -2iT(s \wedge t); \tag{1}$$

where T is the time process; $T(t) = tI$.
- For example if $dr = i\lambda(dP \otimes dQ - dQ \otimes dP)$, $\lambda \in \mathbb{R}$, let

$$\begin{aligned}(\delta_j \otimes \delta'_k)\, r &= i\lambda\left((P(s_j) - P(s_{j-1})) \otimes (Q(t_k) - Q(t_{k-1}))\right. \\ &\quad \left.- (Q(s_j) - Q(s_{j-1})) \otimes (P(t_k) - P(t_{k-1}))\right),\end{aligned}$$

where $s_j = a + \frac{j}{m}(b-a)$, $t_k = c + \frac{k}{n}(d-c)$. Then, by (1), *the* $\left\{(\delta_j \otimes \delta'_k)\, r\right\}_{(j,k) \in \mathbb{N}_m \times \mathbb{N}_n}$ *are weakly commuting.*

- So we can hope to define

$$\prod_{(x,y)\in[a,b[\times[c,d[}(1+dr)=\lim_{m,n\to\infty}\Pi_{(j,k)\in\mathbb{N}_m\times\mathbb{N}_n}\left(I+\left(\delta_j\otimes\delta_k'\right)r\right).$$

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  The approximations obtained by replacing infinitesimals by discrete increments are well-defined.
- In the case $dr = i\lambda\left(dP \otimes dQ - dQ \otimes dP\right)$ we can construct the limit explicitly by recognising $dP \otimes dQ - dQ \otimes dP$ as an infinitesimal *angular momentum*, and hence as the generator of an infinitesimal *rotation*.

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- We write $\left(\delta_j\otimes\delta'_k\right) r = i\lambda\sqrt{\frac{(b-a)(d-c)}{mn}}\left(p_j q'_k - q_j p'_k\right)$ where

$$\begin{aligned} p_j &= \sqrt{m/(b-a)}\left(P(s_j) - P(s_{j-1})\right)\otimes I,\ q_j = \cdots Q\cdots \\ p'_k &= \sqrt{n/(d-c)} I\otimes\left(P(t_k) - P(t_{k-1})\right), q'_k = \cdots Q\cdots \end{aligned}$$

satisfy the standard canonical commutation relations

$$[p_j, q_k] = \left[p'_j, q'_k\right] = -2i\delta_{j,k}, [p_j, p_k] = \left[p_j, p'_k\right] = \left[q_j, p'_k\right] = 0, \text{ etc.}$$

- Using the further approximation, valid for large m, n

$$\begin{aligned} I + \left(\delta_j \otimes \delta_k'\right) r &= I + i\lambda\sqrt{\frac{(b-a)(d-c)}{mn}}\left(p_j q_k' - q_j p_k'\right) \\ &\simeq \exp\left(i\lambda\theta_{m,n}\left(p_j q_k' - q_j p_k'\right)\right) \end{aligned}$$

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where $\theta_{m,n} = \sqrt{\frac{(b-a)(d-c)}{mn}}$,
- and using the standard realisation of the Brownian motions P and Q together with their normalized increments p_j, q_j, p'_k, q'_k in the Fock spaces $\mathcal{F}_a^b = \mathcal{F}\left(L^2\left([a,b[\right)\right)$ and $\mathcal{F}_c^d = \mathcal{F}\left(L^2\left([c,d[\right)\right)$,

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$$\begin{aligned} I+\left(\delta_j\otimes\delta'_k\right)r &= I+i\lambda\sqrt{\frac{(b-a)\,(d-c)}{mn}}\left(p_jq'_k-q_jp'_k\right)\\ &\simeq \exp\left(i\lambda\theta_{m,n}\left(p_jq'_k-q_jp'_k\right)\right)\end{aligned}$$

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- and embedding the canonical orthonormal bases of $\mathbb{C}^m$ and $\mathbb{C}^n$ into $L^2\left([a,b[\right)$ and $L^2\left([c,d[\right)$ as the normalized indicator functions

$$\chi_j=\sqrt{\frac{m}{b-a}}\chi_{[s_{j-1},s_j[},\chi'_k=\sqrt{\frac{n}{d-c}}\chi_{[t_{k-1},t_k[},$$

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- and making the canonical identification

$$\mathcal{F}\left(L^2\left([a,b[\right)\right) \otimes \mathcal{F}\left(L^2\left([c,d[\right)\right) = \mathcal{F}\left(L^2\left([a,b[\right) \oplus L^2\left([c,d[\right)\right)$$

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- and using the rule $\Gamma(UV)=\Gamma(U)\Gamma(V)$ for second quantizations, we approximate the original double product as a discrete double product of of second quantized planar rotations

$$\prod_{[a,b[\times[c,d[}(1+dr)\simeq\Pi_{(j,k)\in\mathbb{N}_m\times\mathbb{N}_n}\Gamma\left(R_{m,n}^{(j,k)}\right)=\Gamma\left(\Pi_{(j,k)\in\mathbb{N}_m\times\mathbb{N}_n}R_{m,n}^{(j,k)}\right)$$

where $R_{m,n}^{(j,k)}$ is the $(m+n)\times(m+n)$ rotation matrix

$$\begin{bmatrix} 1 & \cdots & \overset{(j)}{0} & \cdots & \overset{(m+k)}{0} & \cdots & 0 \\ \vdots & \ddots & \vdots & \cdots & \vdots & \cdots & \vdots \\ {}^{(j)}0 & \cdots & \cos(\lambda\theta_{mn}) & \cdots & -\sin(\lambda\theta_{mn}) & \cdots & 0 \\ \vdots & \cdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ {}^{(m+k)}0 & \cdots & \sin(\lambda\theta_{mn}) & \cdots & \cos(\lambda\theta_{mn}) & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}.$$

We regard $R_{m,n}^{(j,k)}$ and $R_{m,n}=\Pi_{(j,k)\in\mathbb{N}_m\times\mathbb{N}_n}R_{m,n}^{(j,k)}$ as operators on $L^2([a,b[)\oplus L^2([c,d[)$ acting as the identity on the orthogonal complement $((\chi_1,0),(\chi_2,0),\ldots,(\chi_m,0),(0,\chi_1'),(0,\chi_2'),\ldots,(0,\chi_n'))^{\perp}$.

We evaluate $\lim_{m,n\to\infty} R_{m,n}$ in two stages as follows. Setting

$$\alpha = \delta = \cos(\lambda\theta_{mn}), -\beta = \gamma = \sin(\lambda\theta_{mn}),$$

we first consider only the limit as $m \to \infty$ of

$$\prod_{j=1}^{m} \begin{bmatrix} 1 & \cdots & \overset{(j)}{0} & \cdots & \overset{m+1}{0} \\ \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & \cdots & \alpha & \cdots & \beta \\ \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \gamma & \cdots & \delta \end{bmatrix} = \begin{bmatrix} \alpha & \beta\gamma & \cdots & \beta\delta^{m-2}\gamma & \beta\delta^{m-1} \\ 0 & \alpha & \cdots & \beta\delta^{m-3}\gamma & \beta\delta^{m-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \alpha & \beta \\ \gamma & \delta\gamma & \cdots & \delta^{m-1}\gamma & \delta^m \end{bmatrix}$$

as an operator on $L^2([a,b[) \oplus \mathbb{C}$. Since $\theta_{m,n} = \sqrt{\frac{(b-a)(d-c)}{mn}}$,

$$\begin{aligned} \delta^m &= (\cos(\lambda\theta_{mn}))^m \simeq \left(1 - \lambda^2\theta_{mn}^2/2\right)^m = \left(1 - \lambda^2 \frac{(b-a)(d-c)}{2mn}\right)^m \\ &\underset{m\to\infty}{\longrightarrow} \exp\left(-\lambda^2 \frac{(b-a)(d-c)}{2n}\right). \end{aligned}$$

Using this limit we can find the limit of the whole matrix:

$$\left[\begin{array}{cccccc|c} \alpha & \beta\gamma & \beta\delta\gamma & \cdots & \beta\delta^{m-3}\gamma & \beta\delta^{m-2}\gamma & \beta\delta^{m-1} \\ 0 & \alpha & \beta\gamma & \cdots & \beta\delta^{m-4}\gamma & \beta\delta^{m-3}\gamma & \beta\delta^{m-2} \\ 0 & 0 & \alpha & \cdots & \beta\delta^{m-5}\gamma & \beta\delta^{m-4}\gamma & \beta\delta^{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha & \beta\gamma & \beta\delta \\ 0 & 0 & 0 & \cdots & 0 & \alpha & \beta \\ \hline \gamma & \delta\gamma & \delta^2\gamma & \cdots & \delta^{m-2}\gamma & \delta^{m-1}\gamma & \delta^m \end{array}\right]$$

$$\underset{m\to\infty}{\longrightarrow} \left[\begin{array}{cc} I + H_{[a,b[}^{(d-c)/n} & -\left|f_{[a,b[}^{(d-c)/n}\right\rangle \\ \left\langle g_{[a,b[}^{(d-c)/n}\right| & \exp\left(-\lambda^2 \frac{(b-a)(d-c)}{2n}\right) \end{array}\right]$$

where $H_{[a,b[}^{(d-c)/n}$ is the integral operator on $L^2([a,b[)$ whose kernel is

$$(s,t) \mapsto -\lambda^2 \frac{d-c}{n} \chi_{\Delta_{[a,b[}}(s,t)\, e^{-\lambda^2(d-c)(t-s)/(2n)}$$

and the bra and ket (covector and vector) are given by

$$g_{[a,b[}^{(d-c)/n}(s) = e^{-(d-c)\lambda(s-a)/(2n)},\; f_{[a,b[}^{(d-c)/n}(s) = e^{-(d-c)\lambda^2(b-s)/(2n)}.$$

Similarly, we can construct the limit operator on $\mathbb{C}\oplus L^2([a,b[)$ of

$$\prod_{k=1}^{n}\begin{bmatrix} \alpha & .. & \overset{(j)}{\beta} & .. & \overset{n+1}{0} \\ \vdots & \ddots & \vdots & .. & \vdots \\ \gamma & .. & \delta & .. & 0 \\ \vdots & .. & \vdots & \ddots & \vdots \\ 0 & .. & 0 & .. & 1 \end{bmatrix} = \left[\begin{array}{c|cccc} \alpha^n & \beta & \alpha\beta & .. & \alpha^{n-1}\beta \\ \hline \gamma\alpha^{n-1} & \delta & \gamma\beta & .. & \gamma\alpha^{n-2}\beta \\ \gamma\alpha^{n-2} & 0 & \delta & .. & \gamma\alpha^{n-3}\beta \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \gamma & 0 & 0 & .. & \delta \end{array}\right]$$

$$\underset{n\to\infty}{\longrightarrow}\begin{bmatrix} \exp\left(-\lambda^2\frac{(b-a)(d-c)}{2m}\right) & -\left\langle g_{[c.d[}^{(b-a)/m}\right. \\ \left. f_{[c,d[}^{(b-a)/m}\right\rangle & I+H_{[c,d[}^{(b-a)/m} \end{bmatrix}.$$

The matrix product formula is still valid if we replace the scalars α, β, γ and δ by, respectively, an operator, a vector, a covector and a scalar. So we can replace them by the four elements $I+H_{[a,b[}^{(d-c)/n}$, $-\left|f_{[a,b[}^{(d-c)/n}\right\rangle$, $\left\langle g_{[a,b[}^{(d-c)/n}\right|$ and $\exp\left(-\lambda^2\frac{(b-a)(d-c)}{2n}\right)$ of the *earlier* limit operator on $L^2([a,b[)\oplus\mathbb{C}$. When we do this:

- Since the kernel of $H_{[a,b[}^{(d-c)/n}$ is

$$\begin{aligned} (s,t) &\mapsto -\lambda^2 \frac{d-c}{n} \chi_{\Delta_{[a,b[}}(s,t)\, e^{-\lambda^2 (d-c)(t-s)/(2n)} \\ &\simeq -\lambda^2 \frac{d-c}{n} \chi_{\Delta_{[a,b[}}(s,t) + O\left(1/n^2\right), \end{aligned}$$

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- it follows that

$$H_{[a,b[}^{(d-c)/n} \simeq -\left(1/n\right) \lambda^2 \left(d-c\right) V_{[a,b[} + O\left(1/n^2\right)$$

where $V_{[a,b[}$ is the Volterra integral operator whose kernel is $\chi_{\Delta_{[a,b[}}$.

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$$

where $V_{[a,b[}$ is the Volterra integral operator whose kernel is $\chi_{\Delta_{[a,b[}}$.

- Hence when $\alpha = I + H_{[a,b[}^{(d-c)/n}$, at least at a heuristic level,

$$
\begin{aligned}
\alpha^n &= \left(I + H_{[a,b[}^{(d-c)/n}\right)^n \simeq \left(I - \frac{\lambda^2(d-c)}{n} V_{[a,b[}\right)^n \\
&\underset{n\to\infty}{\longrightarrow} \exp\left(-\lambda^2(d-c)\,V_{[a,b[}\right).
\end{aligned}
$$

- Using this limit, we can evaluate explicitly the limit

$$\begin{aligned} & \lim_{n\to\infty}\left\{\lim_{m\to\infty}\Pi_{(j,k)\in\mathbb{N}_m\times\mathbb{N}_n}R_{m,n}^{(j,k)}\right\} \\ = & \lim_{n\to\infty}\left\{\lim_{m\to\infty}\Pi_{1\le k\le n}\left\{\Pi_{1\le j\le m}x_{j,k}\right\}\right\} \\ = & \lim_{n\to\infty}\left\{\Pi_{1\le k\le n}\left\{\lim_{m\to\infty}\Pi_{1\le j\le m}x_{j,k}\right\}\right\}. \end{aligned}$$

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- Similarly, by reversing the order of our limiting procedures we can evaluate

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- Fortunately, the two iterated limits are equal.

$$
\begin{aligned}
\lim_{n\to\infty}\left\{\lim_{m\to\infty}\Pi_{(j,k)\in\mathbb{N}_m\times\mathbb{N}_n}R_{m,n}^{(j,k)}\right\} &= \lim_{m\to\infty}\left\{\lim_{n\to\infty}\Pi_{(j,k)\in\mathbb{N}_m\times\mathbb{N}_n}R_{m,n}^{(j,k)}\right\} \\
&= \begin{bmatrix} I+A_\lambda & B_\lambda \\ C_\lambda & I+D_\lambda \end{bmatrix}
\end{aligned}
$$

where A_λ, B_λ, C_λ and D_λ are respectively integral operators from $L^2([a,b[)$ to itself, from $L^2([c,d[)$ to $L^2([a,b[)$ and vice versa, and from $L^2([c,d[)$ to itself, whose kernels are given repectively by

$$
\begin{aligned}
\mathrm{Ker}\,A_\lambda(s,t) &= \chi_{\Delta_{[a,b[}}(s,t)\sum_{N=0}^{\infty}\frac{(t-s)^N\left(-\lambda^2(d-c)\right)^{N+1}}{N!\,(N+1)!}, \\
\mathrm{Ker}\,B_\lambda(s,t) &= \lambda\chi_{[a,b[\times[c,d[}\sum_{N=0}^{\infty}\frac{\left(-\lambda^2(b-s)(t-c)\right)^N}{(N!)^2}, \\
\mathrm{Ker}\,C_\lambda(s,t) &= -\lambda\chi_{[c,d[\times[a,b[}\sum_{N=0}^{\infty}\frac{\left(-\lambda^2(d-s)(t-a)\right)^N}{(N!)^2}, \\
\mathrm{Ker}\,D_\lambda(s,t) &= \chi_{\Delta_{[c,d[}}(s,t)\sum_{N=0}^{\infty}\frac{(t-s)^N\left(-\lambda^2(b-a)\right)^{N+1}}{N!\,(N+1)!}.
\end{aligned}
$$

- Following all these heuristics, it can then be shown rigorously[2] that $\begin{bmatrix} I+A_\lambda & B_\lambda \\ C_\lambda & I+D_\lambda \end{bmatrix}$ is indeed a unitary operator, and that its second quantization satisfies the quantum stochastic differential equations which rigorously define $\prod_{[a,b[\times[c,d[} (1+i\lambda\,(dP\otimes dQ - dQ\otimes dP))$.

[2] R L Hudson and P Jones, Explicit construction of a unitary double product integral, pp 215-236, in *Noncommutative harmonic analysis with applications to probability III*, eds M Bożeko, A Krystek and L Wojakowski, Banach Center Publications **96** (2012).

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- Thus

$$\prod_{[a,b[\times[c,d[} (1 + i\lambda (dP \otimes dQ - dQ \otimes dP)) = \Gamma\left(\begin{bmatrix} I+A_\lambda & B_\lambda \\ C_\lambda & I+D_\lambda \end{bmatrix}\right).$$

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- The construction of the corresponding triangular double product integral is considerably more difficult because it cannot be evaluated using this iterated limit technique. It is the subject of Yuchen Pei's talk.

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[2] R L Hudson and P Jones, Explicit construction of a unitary double product integral, pp 215-236, in *Noncommutative harmonic analysis with applications to probability III*, eds M Bożeko, A Krystek and L Wojakowski, Banach Center Publications **96** (2012).

- Generally, the triangular and rectanguar product integrals $\prod_{[a,b[\times[c,d[}(1+dr)$ and $\prod_{\Delta_{[a,b[}}(1+dr)$ are unitary [3] if and only if

$$dr + dr^{\dagger} + drdr^{\dagger} = 0. \tag{2}$$

Here the Ito algebra $\mathcal{I} = \mathbb{C}\left(dP, dQ, dT\right)$ has the multiplication table

	dP	dQ	dT
dP	dT	$-idT$	0
dQ	idT	dT	0
dT	0	0	0

and the involution $\dagger$ under which dP, dQ and dT are all self adjoint. Then $\mathcal{I} \otimes \mathcal{I}$ is equipped with the tensor product multiplication and involution.

[3] R L Hudson, Forward and backward adapted quantum stochastic calculus and double product integrals, Russian Journal of Mathematical Physics **21**, No3 (V P Belavkin Memorial Volume), 348-361 (2014).

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- All such dr are easily found. Take arbitrary skew-self-adjoint $d\hat{r}\in\mathcal{I}\otimes\mathcal{I}$. Then $d\hat{r}d\hat{r}^{\dagger} = \xi dT\otimes dT$ for some real ξ and $dr = d\hat{r} - \frac{1}{2}\xi dT\otimes dT$ satisfies (2). Every such dr is found this way.

[3] R L Hudson, Forward and backward adapted quantum stochastic calculus and double product integrals, Russian Journal of Mathematical Physics **21**, No3 (V P Belavkin Memorial Volume), 348-361 (2014).

- Thus, as well as

$$dr = i\left(dP \otimes dQ - dQ \otimes dP\right),$$

$$\begin{aligned} dr' &= i\left(dP \otimes dP + dQ \otimes dQ\right), \\ d\rho &= i\left(dP \otimes dQ + dQ \otimes dP\right) - 2dT \otimes dT, \\ d\rho' &= i\left(dP \otimes dP - dQ \otimes dQ\right) - 2dT \otimes dT \end{aligned}$$

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are all unitary generators.

- Above we found that for the first of these, $\prod_{[a,b[\times[c,d[} (1 + \lambda dr)$ is

$$\Gamma\left(\lim_{m,n\to\infty} \Pi_{(j,k)\in\mathbb{N}_m\times\mathbb{N}_n} \begin{pmatrix} \cos\lambda\theta_{m,n} & -\sin\lambda\theta_{mn} \\ \sin\lambda\theta_{mn} & \cos\lambda\theta_{m,n} \end{pmatrix}_{m,n}^{j,k}\right)$$

$$= \Gamma\begin{pmatrix} I + A_\lambda & B_\lambda \\ C_\lambda & I + D_\lambda \end{pmatrix}$$

for explicitly determined integral operators A_λ, B_λ, C_λ and D_λ.

- For the second type of generator, using the same techniques, we find that $\prod_{[a,b[\times[c,d[} (1 + i\mu (dP \otimes dP + dQ \otimes dQ))$ is

$$\Gamma \left(\lim_{m,n\to\infty} \Pi_{(j,k)\in\mathbb{N}_m\times\mathbb{N}_n} \begin{pmatrix} \cos \mu\theta_{m,n} & i \sin \mu\theta_{m,n} \\ i \sin \mu\theta_{m,n} & \cos \mu\theta_{m,n} \end{pmatrix}^{j,k}_{m,n} \right)$$

$$= \Gamma \begin{pmatrix} I + A'_\mu & B'_\mu \\ C'_\mu & I + D'_\mu \end{pmatrix}.$$

where $A'_\mu = A_\mu$, $B'_\mu = iB_\mu$, $C'_\mu = iC_\mu$ and $D'_\mu = D_\mu$.

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where $A'_\mu = A_\mu$, $B'_\mu = iB_\mu$, $C'_\mu = iC_\mu$ and $D'_\mu = D_\mu$.

- More generally we can consider the real linear combination

$$\begin{aligned} dr_{\lambda,\mu} &= i(\lambda(dP\otimes dQ - dQ\otimes dP) + \mu(dP\otimes dP + dQ\otimes dQ)) \\ &= \lambda dr + \mu dr' \end{aligned}$$

which is also a unitary generator without time correction, since, as well as $dr^2 = dr'^2 = 0$,

$$drdr' = dr'dr = 0.$$

To construct $\prod_{[a,b[\times[c,d[} (1 + dr_{\lambda,\mu})$, we have to find the correct element of $SU(2)$ whose second quantization implements the linear canonical transformation got by conjugating by $\exp(\delta_j \otimes \delta'_k)\, r_{\lambda,\mu}$. Since we know how to implement in this way $dr_{\lambda,0} = \lambda dr$ and $dr_{0,\mu} = \mu dr'$, we can find it as a Trotter product type limit

$$\lim_{N\to\infty}\left(\begin{pmatrix} \cos\frac{\lambda\theta_{m,n}}{N} & -\sin\frac{\lambda\theta_{mn}}{N} \\ \sin\frac{\lambda\theta_{mn}}{N} & \cos\frac{\lambda\theta_{m,n}}{N} \end{pmatrix}\begin{pmatrix} \cos\frac{\mu\theta_{m,n}}{N} & i\sin\frac{\mu\theta_{m,n}}{N} \\ i\sin\frac{\mu\theta_{m,n}}{N} & \cos\frac{\mu\theta_{m,n}}{N} \end{pmatrix}\right)^N$$

$$= \begin{pmatrix} \cos|\nu|\,\theta_{m,n} & -e^{-i\phi}\sin|\nu|\,\theta_{m,n} \\ e^{i\phi}\sin|\nu|\,\theta_{m,n} & \cos|\nu|\,\theta_{m,n} \end{pmatrix}$$

where $\nu = \lambda + i\mu = e^{i\phi}|\nu|$. We find[4]

$$\prod_{[a,b[\times[c,d[} (1 + dr_{\lambda,\mu}) = \Gamma\left(\begin{bmatrix} I + A_{\lambda,\mu} & B_{\lambda,\mu} \\ A_{\lambda,\mu} & I + D_{\lambda,\mu} \end{bmatrix}\right)$$

where now $A_{\lambda,\mu} = A_{|\nu|}$, $B_{\lambda,\mu} = e^{-i\phi}B_{|\nu|}$, $C_{\lambda,\mu} = e^{i\phi}C_{|\nu|}$, $D_{\lambda,\mu} = D_{|\nu|}$.

[4] R L Hudson and Yuchen Pei, Unitary causal quantum stochastic double products as universal interactions I, submitted, Loughborough Mathematics Preprint 2015.

- For the unitary generators

$$\begin{aligned} d\rho_\lambda &= i\lambda\left(dP\otimes dQ + dQ\otimes dP\right) - 2\lambda^2 dT\otimes dT, \\ d\rho'_\mu &= i\mu\left(dP\otimes dP - dQ\otimes dQ\right) - 2\mu^2 dT\otimes dT \end{aligned}$$

the corresponding double product integrals $\prod_{[a,b[\times[c,d[}\left(1+d\rho_\lambda\right)$ and $\prod_{[a,b[\times[c,d[}\left(1+d\rho'_\mu\right)$ are not second quantizations.

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- But they can still be described as unitary implementors of explicit Bogolubov transformations, that is, invertible *real*-linear transformations on $L^2\left([a,b[\right)\oplus L^2\left([c,d[\right)$ which preserve only the imaginary part of the inner product.

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- *Complex* linear Bogolubov transformations are unitaries U; and their second quantizations $\Gamma\left(U\right)$ *implement* them, meaning that $\Gamma\left(U\right)W\left(F\right)\Gamma\left(U\right)^{-1} = W\left(UF\right)$ for each Weyl operator $W\left(F\right)$.

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- Those which are only real-linear can still have unitary implementors in this sense which are not second quantizations..

- When $\rho_\lambda = i\lambda\left(dP \otimes dQ + dQ \otimes dP\right) - 2\lambda^2 dT \otimes dT$[5], we find that each $\exp\left(\delta_j \otimes \delta_k'\right)\rho_\lambda$ unitarily implements the Bogolubov transformation on $\mathbb{C}^2 = \mathbb{R}^4$

$$B = \begin{pmatrix} \cosh\lambda\theta_{m,n} & \kappa\sinh\lambda\theta_{m,n} \\ \sinh\lambda\theta_{m,n}\kappa & \cosh\lambda\theta_{m,n} \end{pmatrix}$$

where κ denotes complex conjugation. This means that

$$\exp\left(\delta_j \otimes \delta_k'\right)\rho_\lambda W\left(\begin{bmatrix} z \\ w \end{bmatrix}\right)\left(\exp\left(\delta_j \otimes \delta_k'\right)\rho_\lambda\right)^{-1} = W\left(B\begin{bmatrix} z \\ w \end{bmatrix}\right)$$

for every corresponding Weyl operator $W\left((z,w)^\tau\right)$.

[5] P Jones, *Unitary double product integrals as implementors of Bogolubov transformations*, Loughborough University PhD thesis (2014).

- When $\rho_\lambda = i\lambda\left(dP \otimes dQ + dQ \otimes dP\right) - 2\lambda^2 dT \otimes dT$[5], we find that each $\exp\left(\delta_j \otimes \delta'_k\right)\rho_\lambda$ unitarily implements the Bogolubov transformation on $\mathbb{C}^2 = \mathbb{R}^4$

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for every corresponding Weyl operator $W\left((z,w)^\tau\right)$.

- Now form the embeddings $B^{j,k}_{m,n}$ as $(m+n) \times (m+n)$ matrices, and embed $\mathbb{C}^{m+n} = \mathbb{C}^m \oplus \mathbb{C}^n$ into $L^2\left([a,b[\right) \oplus L^2\left([c,d[\right)$ as before. Then $\Pi_{(j,k)\in\mathbb{N}_m\times\mathbb{N}_n} B^{j,k}_{m,n}$ is also a Bogolubov transformation and

$$\lim_{m,n\to\infty} \Pi_{(j,k)\in\mathbb{N}_m\times\mathbb{N}_n} B^{j,k}_{m,n} = \begin{bmatrix} I + \mathcal{A}_\lambda & K\mathcal{B}_\lambda \\ \mathcal{C}_\lambda K & I + \mathcal{D}_\lambda \end{bmatrix}.$$

[5] P Jones, *Unitary double product integrals as implementors of Bogolubov transformations*, Loughborough University PhD thesis (2014).

- Here K is functional complex conjugation on $L^2([a,b[)$ and $\mathcal{A}_\lambda$, $\mathcal{B}_\lambda$, $\mathcal{C}_\lambda$, and $\mathcal{D}_\lambda$ are explicit integral operators. Then

$$\prod_{[a,b[\times[c,d[}(1+d\rho_\lambda)\,W\left(\left[\begin{array}{c} f \\ g \end{array}\right]\right)\left(\prod_{[a,b[\times[c,d[}(1+d\rho_\lambda)\right)^{-1} = W\left(\left[\begin{array}{cc} I+\mathcal{A}_\lambda & K\mathcal{B}_\lambda \\ \mathcal{C}_\lambda K & I+\mathcal{D}_\lambda \end{array}\right]\left[\begin{array}{c} f \\ g \end{array}\right]\right) \quad (3)$$

for every Weyl operator $W\left((f,g)^\tau\right)$.

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$$\prod_{[a,b[\times[c,d[} (1+d\rho_\lambda)\, W\left(\begin{bmatrix} f \\ g \end{bmatrix}\right)\left(\prod_{[a,b[\times[c,d[} (1+d\rho_\lambda)\right)^{-1} = W\left(\begin{bmatrix} I+\mathcal{A}_\lambda & K\mathcal{B}_\lambda \\ \mathcal{C}_\lambda K & I+\mathcal{D}_\lambda \end{bmatrix}\begin{bmatrix} f \\ g \end{bmatrix}\right) \qquad (3)$$

  for every Weyl operator $W\left((f,g)^\tau\right)$.
- $\prod_{[a,b[\times[c,d[} (1+d\rho_\lambda)$ is not uniquely characterized by (3). Any unimodular complex multiple will do the same.

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  for every Weyl operator $W\left((f,g)^\tau\right)$.
- $\prod_{[a,b[\times[c,d[}(1+d\rho_\lambda)$ is not uniquely characterized by (3). Any unimodular complex multiple will do the same.
- But there is a unique *system* of implementors which is separately evolutionary in $[a,b[$ and $[c,d[$, and covariant under shifts and time reversal. When defined as a solution of qsde's, $\prod_{[a,b[\times[c,d[}(1+d\rho_\lambda)$ is known to have these properties. So $\left(\prod_{[a,b[\times[c,d[}(1+d\rho_\lambda)\right)_{a\leq b,c\leq d}$ is the unique bievolutionary bicovariant *system* of unitary implementors of the corresponding system of explicit Bogolubov transformations.

In the case of $d\rho'_\mu = i\mu\left(dP \otimes dP - dQ \otimes dQ\right) - 2\mu^2 dT \otimes dT$ we replace

$$B_\lambda = \begin{pmatrix} \cosh \lambda\theta_{m,n} & \sinh \lambda\theta_{m,n}\kappa \\ \sinh \lambda\theta_{m,n}\kappa & \cosh \lambda\theta_{m,n} \end{pmatrix}$$

by

$$B'_\mu = \begin{pmatrix} \cosh \mu\theta_{m,n} & -i \sinh \mu\theta_{m,n}\kappa \\ i \sinh \mu\theta_{m,n}\kappa & \cosh \mu\theta_{m,n} \end{pmatrix}$$

and similarly costruct the explicit Bogolubov transformation

$$\lim_{m,n\to\infty} \Pi_{(j,k)\in\mathbb{N}_m\times\mathbb{N}_n} B'^{j,k}_{\mu,m,n} = \begin{bmatrix} I + \mathcal{A}'_\mu & K\mathcal{B}'_\mu \\ \mathcal{C}'_\mu K & I + \mathcal{D}'_\mu \end{bmatrix}$$

for which

$$\prod_{[a,b[\times[c,d[]} \left(1 + i\mu\left(dP \otimes dP - dQ \otimes dQ\right) - 2\mu^2 dT \otimes dT\right)$$

is the unique bievolutionary bicovariant system of unitary implementors.

- The combination

$$\begin{aligned} d\rho_{\lambda,\mu} &= i\{\lambda\,(dP\otimes dQ + dQ\otimes dP) \\ &\quad \mu\,(dP\otimes dP - dQ\otimes dQ) - 2\,|\lambda + i\mu|^2\, dT\otimes dT\Big\} \end{aligned}$$

is also a unitary generator, since

$$\begin{aligned} (dP\otimes dQ + dQ\otimes dP)\,(dP\otimes dP - dQ\otimes dQ) &= 4i dT\otimes dT, \\ (dP\otimes dP - dQ\otimes dQ)\,(dP\otimes dQ + dQ\otimes dP) &= -4i dT\otimes dT. \end{aligned}$$

- The combination

$$d\rho_{\lambda,\mu} = i\{\lambda(dP\otimes dQ + dQ\otimes dP) \\ \mu(dP\otimes dP - dQ\otimes dQ) - 2|\lambda+i\mu|^2 dT\otimes dT\}$$

is also a unitary generator, since

$$\begin{aligned}(dP\otimes dQ + dQ\otimes dP)(dP\otimes dP - dQ\otimes dQ) &= 4idT\otimes dT, \\ (dP\otimes dP - dQ\otimes dQ)(dP\otimes dQ + dQ\otimes dP) &= -4idT\otimes dT.\end{aligned}$$

- To construct the corresponding product integral we again apply the Trotter product trick:

$$\left\{\begin{bmatrix} \cosh\frac{\lambda}{N}\theta_{m,n} & \sinh\frac{\lambda}{N}\theta_{m,n}\kappa \\ \sinh\frac{\lambda}{N}\theta_{m,n}\kappa & \cosh\frac{\lambda}{N}\theta_{m,n}\end{bmatrix}\begin{bmatrix} \cosh\frac{\mu}{N}\theta_{m,n} & -i\sinh\frac{\mu}{N}\theta_{m,n}\kappa \\ i\sinh\frac{\mu}{N}\theta_{m,n}\kappa & \cosh\frac{\mu}{N}\theta_{m,n}\end{bmatrix}\right\}^N$$

$$\underset{N\to\infty}{\longrightarrow}\begin{bmatrix} \cosh|\nu|\theta_{m,n} & e^{-i\phi}\kappa\sinh|\nu|\theta_{m,n} \\ e^{i\phi}\kappa\sinh|\nu|\theta_{m,n} & \cosh|\nu|\theta_{m,n}\end{bmatrix},$$

where as before $\nu = \lambda + i\mu = |\nu| e^{i\phi}$.

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- The resulting real-linear operator is $\begin{bmatrix} I + \mathcal{A}_{\lambda,\mu} & K\mathcal{B}_{\lambda\mu} \\ \mathcal{C}_{\lambda\mu}K & I + \mathcal{D}_{\lambda\mu} \end{bmatrix}$, where $\mathcal{A}_{\lambda,\mu}$, $\mathcal{B}_{\lambda,\mu}$, $\mathcal{C}_{\lambda,\mu}$ and $\mathcal{D}_{\lambda,\mu}$, are integral operators with explicit kernels. It is a Bogolubov transformation. The unique bievolutionary covariant system of unitary implementors of these transformations gives the corresponding family of rectangular product integrals.

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- It is possible to use the Trotter product trick twice over to construct the general unitary double product, with generator of form $dr_{\lambda,\mu} + d\rho_{\lambda',\mu'} + \eta dT \otimes dT$.

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- it is (*possibly*) more interesting to look at the unitary generator

$$\begin{aligned} & i\lambda dP \otimes dQ - \lambda^2 dT \otimes dT \\ = \; & \frac{1}{2} i\lambda \left(dP \otimes dQ - dQ \otimes dP \right) + \frac{1}{2} i \left(\lambda \left(dP \otimes dQ + dQ \otimes dP \right) \right) \\ & -2\lambda^2 dT \otimes dT. \end{aligned}$$

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- $$\left(\begin{bmatrix} \cosh \lambda\theta_{mn}/n & -i\kappa \sinh \lambda\theta_{mn}/n \\ i\kappa \sinh \lambda\theta_{mn}/n & \cosh \lambda\theta_{mn}/n \end{bmatrix} \begin{bmatrix} \cos \lambda\theta_{mn}/n & -\sin \lambda\theta_{mn}/n \\ \sin \lambda\theta_{mn}/n & \cos \lambda\theta_{mn}/n \end{bmatrix}\right.$$
$$\simeq \left(\begin{bmatrix} 1 & -i\kappa\lambda\theta_{mn}/n \\ i\kappa\lambda\theta_{mn}/n & 1 \end{bmatrix} \begin{bmatrix} 1 & -\lambda\theta_{mn}/n \\ \lambda\theta_{mn}/n & 1 \end{bmatrix}\right)^n$$

- $$\simeq \begin{bmatrix} 1 & -(i\kappa+1)\lambda\theta_{mn}/n \\ (i\kappa+1)\lambda\theta_{mn}/n & 1 \end{bmatrix}^n \simeq \exp\{(i\kappa+1)\lambda\theta_{mn}J\}$$
  where $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

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  where $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.
- Since $J^2 = -I$

  $$\exp\{(i\kappa+1)\lambda\theta_{mn}J\} = \cos\{(i\kappa+1)\lambda\theta_{mn}\} + \sin\{(i\kappa+1)\lambda\theta_{mn}\}J$$

  $$\cos((i\kappa+1)\lambda\theta_{mn}) = \cos(i\kappa\lambda\theta_{mn})\cos\lambda\theta_{mn} - \sin(i\kappa\lambda\theta_{mn})\sin\lambda\theta_{mn},$$

  $$\cos(i\kappa\lambda\theta_{mn}) = 1 - \frac{i\kappa\lambda\theta_{mn}i\kappa\lambda\theta_{mn}}{2!} + \cdots = 1 - \frac{\lambda^2\theta_{mn}^2}{2!} + \cdots = \cos\lambda\theta_{mn}$$

  $$\sin(i\kappa\lambda\theta_{mn}) = i\kappa\lambda\theta_{mn} - i\kappa\frac{\lambda^3\theta_{mn}^3}{3!} + i\kappa\frac{\lambda\theta_{mn}^5}{5!} - \cdots = i\kappa\sin\lambda\theta_{mn}.$$

  So

  $$\begin{aligned} \cos((i\kappa+1)\lambda\theta_{mn}) &= (\cos\lambda\theta_{mn})^2 + i\kappa(\sin\lambda\theta_{mn})^2, \\ \sin((i\kappa+1)\lambda\theta_{mn}) &= i\kappa\sin\lambda\theta_{mn}\cos\lambda\theta_{mn} + \cos\lambda\theta_{mn}\sin\lambda\theta_{mn}. \end{aligned}$$

So

$$\begin{aligned}\exp\left\{(i\kappa+1)\lambda\theta_{mn}J\right\} &= \cos\left((i\kappa+1)\lambda\theta_{mn}\right)I+\sin\left((i\kappa+1)\lambda\theta_{mn}\right)J \\ &= \begin{bmatrix}\alpha & \beta \\ \gamma & \delta\end{bmatrix}\end{aligned}$$

where

$$\begin{aligned}\alpha &= \delta=\left(\cos\lambda\theta_{mn}\right)^2+i\kappa\left(\sin\lambda\theta_{mn}\right)^2 \\ -\beta &= \gamma=i\kappa\sin\lambda\theta_{mn}\cos\lambda\theta_{mn}+\cos\lambda\theta_{mn}\sin\lambda\theta_{mn},\end{aligned}$$

Then for large m

$$\begin{aligned}\alpha^m &\simeq \left(1-(1+i\kappa)\lambda^2\frac{(b-a)(d-c)}{mn}\right)^m \\ &\simeq e^{-(1+i\kappa)\lambda^2(b-a)(d-c)/n} \\ &= e^{-\lambda^2(b-a)(d-c)/n}+\cos\left(\lambda^2(b-a)(d-c)/n\right) \\ &\quad +i\kappa\sin\left(\lambda^2(b-a)(d-c)/n\right)\end{aligned}$$

So we may play the same iterated limit game as before.

Thank you for listening.