The Connes Embedding Problem and Quantum Complexity Theory

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A Type classification of von Neumann algebras

1) Give an introduction to von Neumann algebras by proving the bicommutant theorem and go on to define factors and the central support of an element.

   (This is Ch. 3 up to and including Par. 3.9 in [SZ19].)

   1 talk

2) Introduce, discuss and prove the type classification of von Neumann algebras.

   (This is Ch. 4 up to Cor. 4.20 in [SZ19]. The main result is Thm. 4.17 therein.)

   2 talks

B The Connes embedding problem

3) Define hyperfinite von Neumann algebras and construct a model for the hyperfinite II_1-factor as explained in [CP15, Sec. 2.3].

   (You will also need to recall some stuff from [CP15, Appx. A.3] for this.)

   1 talk

4) Formulate and explain the Connes embedding problem.

   (You can use [CP15, Sec. 3.1 & Appx. A.4] as a reference. You will also need to recall the GNS representation associated to a state on the C*-algebra.)

   1 talk

C Group von Neumann algebras

5) Construct the group von Neumann algebras; and prove that if the group has infinite conjugacy classes, then its group von Neumann algebra is a II_1-factor.

   (This is Sec. 1.4.2 up to and including Rem. 1.4.13 in [CP15].)

   1 talk

6) Recall completely positive maps, operator systems, Stinepring dilations and multiplicative domains.

   (This is Sec. 1.5 up to Defn. 1.5.8 in [BO08].)

   0.5 talks

7) Discuss the enveloping von Neumann algebra of a C*-algebra, and especially explain [Bla06, Cor. III.5.2.7].

(The predual M_* is defined at the beginning of Sec. III.2 of [Bla06]. You will probably need to discuss some stuff from Sec. III.2.1, like III.2.1.8., therein.)

0.5 talks

8) Define conditional expectations and discuss the proof of Tomiyama's theorem [BO08, Thm 1.5.10].

   0.5 talks

9) Recall Arveson's extension theorem (without proof) [BO08, Thm 1.6.1] and use it to show that injective C*-algebras are exactly those which are the image of a conditional expectation on $\mathcal{B}(H)$ ([SS08, 2nd paragraph of Sec. 3.8]).

   Prove that the group von Neumann algebra is injective if and only if the group is amenable [SS08, Thm. 3.8.2].[1]

   0.5 talks

D Kirchberg's QWEP conjecture

10) Discuss the problem of tensor products of C*-algebras and explain the construction of both the minimal (i.e. spatial) and maximal ones [BO08, Sec. 3.3].

    Prove [BO08, Props. 3.6.5 & 3.6.6], define the WEP [BO08, Defn. 3.6.7] and conclude [BO08, Cor. 3.6.8].

    1 talk

11) Discuss the (local) liftig property (L)LP [BO08, Sec. 13.1].

    1 talk

12) Discuss the interplay between the LLP and the WEP [BO08, Sec. 13.2].

    1 talk

13) Discuss the equivalence of the Connes Embedding Problem with Kirchberg's QWEP conjecture [BO08, Sec. 13.3].

    (You can ignore Statement (4) in [BO08, Thm. 13.3.1]. To stay in time with your talk, you can leave out the proofs of one or two of the lemmas in this section. In the actual proof of Thm. 13.3.1 some results from Section 6, resp. Section 9 of [BO08] are used; you will need to blackbox them, though try to explain them as well as time permits.)

    2 talks

E Tsirelson's problem

14) Introduce Tsirelson's problem and prove that the QWEP conjecture implies it.

    (These are Sections 2 & 3 and Thm. 4.1 in [Fri]. It seems that we do not need the second part of Sec. 3, i.e. the stuff after the proof of Prop. 3.4 therein.)

    1 talk

15) Prove the converse to the above, i.e. that Tsirelson's problem implies the QWEP conjecture [Fri, Sec. 5].

    2 talks

[1] The reference says *hyperfinite* instead of *injective*, but the proof shows/uses injectivity. That hyperfiniteness and injectivity are equivalent will be used as a black box in this seminar.

F Quantum complexity theory

16) Discuss basic notions from complexity theory, define the complexity class MIP, and discuss non-local games [Gol, Sec. 4].

    (Sec. 4 in [Gol] discusses a lot of irrelevant stuff, e.g. all the different complexity classes. We crucially need the definition of MIP and the stuff in the Subsec. 4.4.)

    0.5 talks

17) Discuss the complexity classes MIP* and RE [Gol, Sec. 5].

    1 talk

18) Discuss Tsirelson's problem and its relation to MIP*=RE [Gol, Sec. 6.1 & 6.2].

    (Thm. 6.2 of [Gol] will be proven, though with other notation, in Part E of this seminar.)

    0.5 talks

References

[Bla06] B. Blackadar. *Operator Algebras, Theory of C*-Algebras and von Neumann Algebras.* Encyclopaedia of Mathematical Sciences, Vol. 122, Operator Algebras and Non-Commutative Geometry III. Springer, 2006. DOI: 10.1007/3-540-28517-2.

[BO08] N. P. Brown and N. Ozawa. *C*-algebras and finite-dimensional approximations.* Vol. 8. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2008. DOI: 10.1090/gsm/088.

[CP15] J. P. G. do Carmo Paulos. "The hyperfinite II_1 factor and Connes Embedding conjecture". PhD thesis. Instituto Superior Técnico, Universidade de Lisboa, 2015.

[Fri] T. Fritz. "Tsirelson's problem and Kirchberg's conjecture". arXiv:1008.1168.

[Gol] I. Goldbring. "The Connes Embedding Problem: A guided tour". arXiv:2109.12682.

[SS08] A. M. Sinclair and R. R. Smith. *Finite von Neumann Algebras and Masas.* LMS Lecture Note Series, vol. 351. Cambridge University Press, 2008.

[SZ19] Ş. V. Strătilă and L. Zsidó. *Lectures on von Neumann Algebras.* 2nd ed. Cambridge–IISc Series. Cambridge University Press, 2019.