Relativizations of the P =? DNP Question over the Complex Numbers

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The machines

Computation instructions:

$$l:\ Z_i := Z_j \circ Z_k, \qquad \circ \in \{+, -, \cdot\},$$

$$l:\ Z_j := c,$$

Branching instructions:

$$l:\ \text{if } Z_j = 0 \text{ then goto } l_1 \text{ else goto } l_2,$$

Copy instructions:

$$l:\ Z_{I_j} := Z_{I_k},$$

Index instructions:

$$l:\ I_j := 1,$$

$$l:\ I_j := I_j + 1,$$

$$l:\ \text{if } I_j = I_k \text{ then goto } l_1 \text{ else goto } l_2.$$

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The complexity classes

The input: $\mathbf{x} = (x_1, \ldots, x_n) \in \bigcup_{i \geq 1} \mathbb{C}^i$.

The guesses: $\mathbf{y} = (y_1, \ldots, y_m) \in \bigcup_{i \geq 1} \mathbb{C}^i$.

Assignment: $\mathbf{x} \mapsto Z_1, \ldots, Z_n \quad \mathbf{y} \mapsto Z_{n+1}, \ldots, Z_{n+m} \quad n \mapsto I_1$

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Polynomial time: $\mathrm{cost}(\mathbf{x}) \leq kn^c$.

size($\mathbf{x}$): n.

cost($\mathbf{x}$): Number of executed instructions on $\mathbf{x}$.

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$\underbrace{y_1, \ldots, y_m = 0}$ $\Downarrow$ $\mathrm{P}_{\mathbb{C}}$	$\underbrace{y_1, \ldots, y_m \in \{0, 1\}}$ $\Downarrow$ $\mathrm{DNP}_{\mathbb{C}}$	$\underbrace{y_1, \ldots, y_m \in \mathbb{C}}$ $\Downarrow$ $\mathrm{NP}_{\mathbb{C}}$

The oracle machines

An oracle: $\mathcal{O} \subseteq \mathbb{C}^\infty$.

The oracle machines:

$$\text{if } (Z_1, \ldots, Z_{I_1}) \in \mathcal{O} \text{ then goto } l_1 \text{ else goto } l_2.$$

$\Rightarrow$

$$\mathrm{P}_{\mathbb{C}} \subseteq \mathrm{DNP}_{\mathbb{C}} \subseteq \mathrm{NP}_{\mathbb{C}}.$$

$$\mathrm{P}_{\mathbb{C}}^{\mathcal{O}} \subseteq \mathrm{DNP}_{\mathbb{C}}^{\mathcal{O}} \subseteq \mathrm{NP}_{\mathbb{C}}^{\mathcal{O}}.$$

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A summary (1)

Structure	$P \neq DNP$	$DNP \neq NP$	$P^Q \neq DNP^Q$	$DNP^Q \neq NP^Q$
$(\{0,1\}; 0, 1; ; =)$	?	no		
$(\mathbb{Z}; \mathbb{Z}; +, -; =)$	yes	yes		
$(\mathbb{R}; \mathbb{R}; +, -; =)$	yes	no		
$(\mathbb{R}; \mathbb{R}; \cdot, +, -; =)$	?	yes		
$(\mathbb{C}; \mathbb{C}; \cdot, +, -; =)$	?	?		

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here: derived from $DNP_{\mathbb{R}}^{O}$

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by analogy with $\mathbb{R}$

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Natural problems as oracles (1)

Analogously to $(\mathbb{R}; \mathbb{R}; \cdot, +, -; =)$ we get (1), (2), and (3).

(1) $\mathbb{Q} \in \mathrm{NP}_{\mathbb{C}}^{\mathbb{Z}} \setminus \mathrm{DNP}_{\mathbb{C}}^{\mathbb{Z}}$,

(2) $\mathbb{Q}^2 \in \mathrm{NP}_{\mathbb{C}}^{\mathbb{Q}} \setminus \mathrm{DNP}_{\mathbb{C}}^{\mathbb{Q}}$ where $\mathbb{Q}^2 = \{q^2 \mid q \in \mathbb{Q}\}$,

(3) $\mathbb{R}_+ \in \mathrm{NP}_{\mathbb{C}}^{\mathbb{R}} \setminus \mathrm{DNP}_{\mathbb{C}}^{\mathbb{R}}$ where $\mathbb{R}_+ = \{r \mid r \in \mathbb{R} \;\&\; r > 0\}$,

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(4) $\mathbb{D} \in \mathrm{NP}_{\mathbb{C}}^{\mathbb{R}_+} \setminus \mathrm{DNP}_{\mathbb{C}}^{\mathbb{R}_+}$ where $\mathbb{D} = \{(c,l) \in \mathbb{C} \times \mathbb{R}_+ \mid |c| = l\}$.

We want to show (4).

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Natural problems as oracles (2a)

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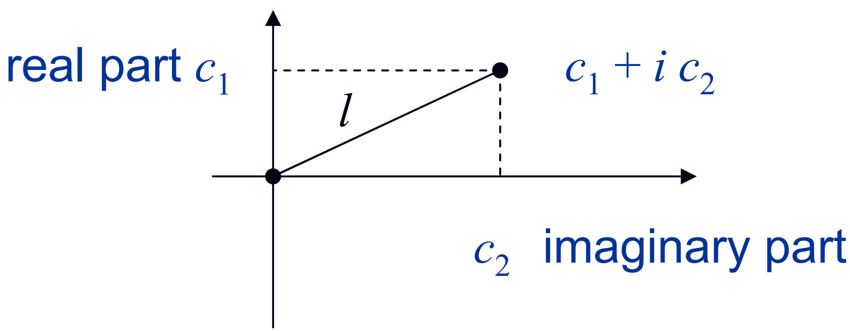

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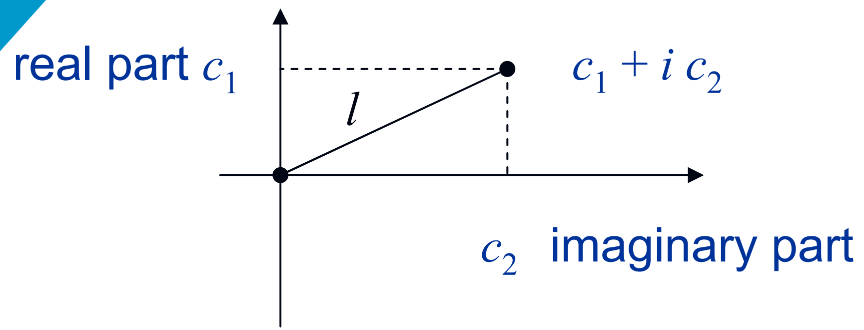
an NP machine can guess c_1, c_2



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$$\Rightarrow \mathbb{D} \in \mathrm{NP}_{\mathbb{C}}^{\mathbb{R}_+}$$

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Natural problems as oracles (2b)

Assume, $\mathbb{D} = \{(c, l) \in \mathbb{C} \times \mathbb{R}_+ \mid |c| = l\} \in \mathrm{DNP}_{\mathbb{C}}^{\mathbb{R}_+}$.

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each computation path of $\mathcal{N}$ on $(x, 1)$ can be described by some

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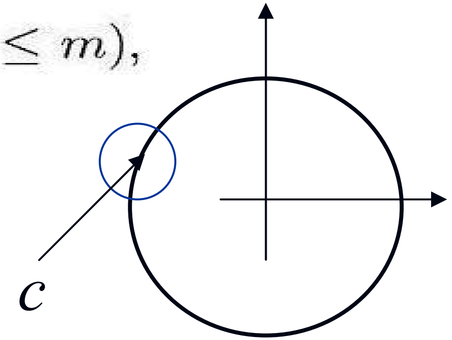

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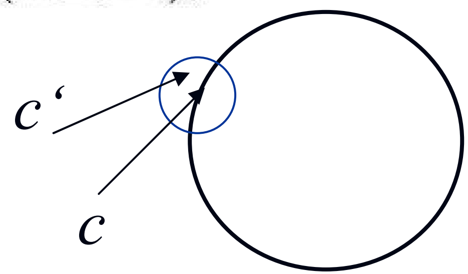

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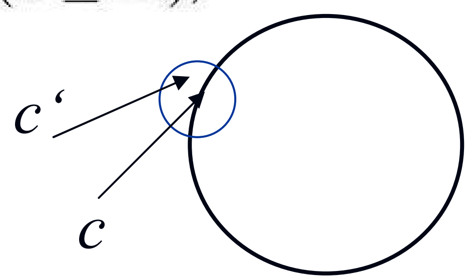

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$\Rightarrow$ $(c, 1)$ and $(c', 1)$ have the same computation path.

$\Rightarrow$ $\mathbb{D} \notin \mathrm{DNP}_{\mathbb{C}}^{\mathbb{R}_+}$

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An oracle Q with $\mathrm{DNP}_{\mathbb{C}}^{Q} \neq \mathrm{NP}_{\mathbb{C}}^{Q}$

$U \subseteq \mathbb{C}^\infty$	set of codes
$\mathbf{u} \in U$	code of the pair $(p_\mathbf{u}, P_\mathbf{u})$
$p_\mathbf{u}$	polynomial
$P_\mathbf{u}$	program of a $\mathrm{DNP}^{\mathcal{O}}$-machine.
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derived from the Emerson oracle

$V_0 = \emptyset$. Stage $i \geq 1$:

$$K_i = \{\mathbf{u} \in U \mid (\forall \mathcal{B} \subseteq \mathbb{C}^\infty)$$
$$(\mathcal{N}_\mathbf{u}^\mathcal{B} \text{ does not use any integer } j > i \text{ in a query on input } \mathbf{u})\},$$

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$$(\mathcal{N}_\mathbf{u}^{\mathcal{B}} \text{ does not use any integer } j > i \text{ in a query on input } \mathbf{u})\},$$
$$W_i = \cup_{k<i} V_k,$$
$$V_i = \{(i+1, \mathbf{u}) \mid \mathbf{u} \in K_i \ \& \ \mathcal{N}_\mathbf{u}^{W_i} \text{ does not accept } \mathbf{u}\}.$$

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$V_0 = \emptyset$. Stage $i \geq 1$:

$$K_i = \{\mathbf{u} \in U \mid (\forall \mathcal{B} \subseteq \mathbb{C}^\infty)$$
$$(\mathcal{N}_\mathbf{u}^{\mathcal{B}} \text{ does not use any integer } j > i \text{ in a query on input } \mathbf{u})\},$$

$$W_i = \cup_{k<i} V_k,$$

$$V_i = \{(i+1, \mathbf{u}) \mid \mathbf{u} \in K_i \;\&\; \mathcal{N}_\mathbf{u}^{W_i} \text{ does not accept } \mathbf{u}\}.$$

$$\mathcal{Q}_1 = \cup_{i \geq 1} W_i.$$

$$L_1 = \{\mathbf{y} \mid (\exists n \in \mathbb{N}^+)((n, \mathbf{y}) \in \mathcal{Q}_1)\} \in \mathrm{NP}_{\mathbb{C}}^{\mathcal{Q}_1} \setminus \mathrm{DNP}_{\mathbb{C}}^{\mathcal{Q}_1}.$$

A summary (3)

Structure	$P \neq DNP$	$DNP \neq NP$	$P^{\mathcal{Q}} \neq DNP^{\mathcal{Q}}$	$DNP^{\mathcal{Q}} \neq NP^{\mathcal{Q}}$
$(\{0,1\}; 0,1; ; =)$	?	no	the Baker-Gill-Solovay oracle	no oracle
$(\mathbb{Z}; \mathbb{Z}; +, -; =)$	yes	yes	$\emptyset$	$\emptyset$
$(\mathbb{R}; \mathbb{R}; +, -; =)$	yes	no	$\emptyset$	the Emerson oracle
$(\mathbb{R}; \mathbb{R}; \cdot, +, -; =)$	?	yes		$\emptyset$
$(\mathbb{C}; \mathbb{C}; \cdot, +, -; =)$	?	?		$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}_+$ an oracle $\mathcal{Q}_1$ derived from the Eme. oracle

A summary (4)

Structure	$P \neq DNP$	$DNP \neq NP$	$P^{\mathcal{Q}} \neq DNP^{\mathcal{Q}}$	$DNP^{\mathcal{Q}} \neq NP^{\mathcal{Q}}$
$(\{0,1\}; 0,1; ; =)$	?	no	the Baker-Gill-Solovay oracle	no oracle
$(\mathbb{Z}; \mathbb{Z}; +, -; =)$	yes	yes	$\emptyset$	$\emptyset$
$(\mathbb{R}; \mathbb{R}; +, -; =)$	yes	no	$\emptyset$	the Emerson oracle
$(\mathbb{R}; \mathbb{R}; \cdot, +, -; =)$	?	yes	?	$\emptyset$
$(\mathbb{C}; \mathbb{C}; \cdot, +, -; =)$	?	?	?	$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}_+$ an oracle $\mathcal{Q}_1$ derived from the Eme. oracle

An oracle Q with $P_C^{\,Q} \neq DNP_C^{\,Q}$

$i \in \mathbb{N}^+$	the code of a pair (p_i, P_i)
p_i	polynomial
P_i	program of a $\mathrm{P}^{\mathcal{O}}$-machine using only the constants 0 and 1
$\mathcal{N}_i^{\mathcal{B}}$	machine using $\mathcal{B} \subseteq \mathbb{C}^\infty$, P_i, and the time bound p_i

An oracle Q with $P_C^Q \neq DNP_C^Q$

$i \in \mathbb{N}^+$	the code of a pair (p_i, P_i)
p_i	polynomial
P_i	program of a $P^{\mathcal{O}}$-machine using only the constants 0 and 1
$\mathcal{N}_i^{\mathcal{B}}$	machine using $\mathcal{B} \subseteq \mathbb{C}^\infty$, P_i, and the time bound p_i

$V_0 = \emptyset$, $m_0 = 0$. Stage $i \geq 1$: Let $n_i > m_{i-1}$ and $p_i(n_i) + n_i < 2^{n_i}$.

$$W_i = \bigcup_{j<i} V_j,$$

$$V_i = \{\mathbf{x} \in \{0,1\}^{n_i} \mid \mathcal{N}_i^{W_i} \text{ rejects } (0,\ldots,0) \in \mathbb{C}^{n_i} \;\&\; \mathbf{x} \text{ is not queried by } \mathcal{N}_i^{W_i} \text{ on input } (0,\ldots,0) \in \mathbb{C}^{n_i}\}$$

$$m_i = 2^{n_i}.$$

An oracle Q with $P_C^Q \neq DNP_C^Q$

$i \in \mathbb{N}^+$	the code of a pair (p_i, P_i)
p_i	polynomial
P_i	program of a $\mathrm{P}^{\mathcal{O}}$-machine using only the constants 0 and 1
$\mathcal{N}_i^{\mathcal{B}}$	machine using $\mathcal{B} \subseteq \mathbb{C}^\infty$, P_i, and the time bound p_i

$V_0 = \emptyset$, $m_0 = 0$. Stage $i \geq 1$: Let $n_i > m_{i-1}$ and $p_i(n_i) + n_i < 2^{n_i}$.

$$W_i = \bigcup_{j<i} V_j,$$
$$V_i = \{\mathbf{x} \in \{0,1\}^{n_i} \mid \mathcal{N}_i^{W_i} \text{ rejects } (0,\ldots,0) \in \mathbb{C}^{n_i} \;\&\; \mathbf{x} \text{ is not queried by } \mathcal{N}_i^{W_i} \text{ on input } (0,\ldots,0) \in \mathbb{C}^{n_i}\}$$
$$m_i = 2^{n_i}.$$

$$\mathcal{Q}_C = \bigcup_{i\geq 1} W_i,$$
$$L_C = \{\mathbf{y} \mid (\exists i \in \mathbb{N}^+)(\mathbf{y} \in \mathbb{C}^{n_i} \;\&\; V_i \neq \emptyset)\} \in \mathrm{DNP}_C^{\mathcal{Q}_C} \setminus \mathrm{P}_C^{\mathcal{Q}_C}.$$

by analogy with the BGS oracle

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An oracle Q with $\mathrm{P}_{\mathbb{C}}^{Q} \neq \mathrm{DNP}_{\mathbb{C}}^{Q}$?

? ? ?

$i \in \mathbb{N}^+$	the code of a pair (p_i, P_i)
p_i	polynomial
P_i	program of a $\mathrm{P}^{\mathcal{O}}$-machine using the constants $c_1, \ldots, c_{k_i}$
$1, \ldots, k_i$	codes of $c_1, \ldots, c_{k_i} \in \mathbb{C}$
$\mathcal{N}_i^{\mathcal{B}, c_1, \ldots, c_{k_i}}$	uses $\mathcal{B} \subseteq \mathbb{C}^\infty$, $c_1, \ldots, c_{k_i}$, P_i, and the time bound p_i

$$V_i = \{\mathbf{x} \in \{0,1\}^{n_i} \mid \forall c_1 \cdots \forall c_{k_i} (\mathcal{N}_i^{W_i, c_1, \ldots, c_{k_i}} \text{ rejects } (0, \ldots, 0) \in \mathbb{C}^{n_i} \;\&\; \mathbf{x} \text{ is not queried by } \mathcal{N}_i^{W_i, c_1, \ldots, c_{k_i}} \text{ on } (0, \ldots, 0) \in \mathbb{C}^{n_i})\}$$

? ? ?

$\Rightarrow$

$$\exists j \forall c_1 \cdots \forall c_{k_j} (\mathcal{N}_j^{W_j, c_1, \ldots, c_{k_j}} \text{ rejects } (0, \ldots, 0) \in \mathbb{C}^{n_j} \;\&\; V_j = \emptyset).$$

An oracle Q with $P_{\mathbb{C}}^{Q} \neq DNP_{\mathbb{C}}^{Q}$

$\sum_{j_1,\ldots,j_k \geq 0} \alpha_{j_1,\ldots,j_k} c_1^{j_1} \cdots c_k^{j_k} \in \{0,1\}$?

where $k \leq k_i$, $c_1, \ldots, c_{k_i} \in \mathbb{C}$, $\alpha_{j_1,\ldots,j_k} \in \mathbb{Z}$.

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An oracle Q with $\mathrm{P}_{\mathbb{C}}^{Q} \neq \mathrm{DNP}_{\mathbb{C}}^{Q}$

$$\sum_{j_1,\ldots,j_k \geq 0} \alpha_{j_1,\ldots,j_k} c_1^{j_1} \cdots c_k^{j_k} \in \{0,1\}?$$

where $k \leq k_i$, $c_1, \ldots, c_{k_i} \in \mathbb{C}$, $\alpha_{j_1,\ldots,j_k} \in \mathbb{Z}$.

$$\begin{array}{l} p(c_k) = 0? \\ p(c_k) = 1? \end{array} \qquad \text{for} \quad p(x) = \sum_{j_1,\ldots,j_k \geq 0} \alpha_{j_1,\ldots,j_k} c_1^{j_1} \cdots c_{k-1}^{j_{k-1}} x^{j_k}.$$

An oracle Q with $\mathrm{P}_{\mathbb{C}}^{Q} \neq \mathrm{DNP}_{\mathbb{C}}^{Q}$

$$\sum_{j_1,\ldots,j_k\geq 0} \alpha_{j_1,\ldots,j_k} c_1^{j_1}\cdots c_k^{j_k} \in \{0,1\}?$$

where $k \leq k_i$, $c_1,\ldots,c_{k_i} \in \mathbb{C}$, $\alpha_{j_1,\ldots,j_k} \in \mathbb{Z}$.

$$\begin{array}{l} p(c_k)=0? \\ p(c_k)=1? \end{array} \quad \text{for} \quad p(x)=\sum_{j_1,\ldots,j_k\geq 0} \alpha_{j_1,\ldots,j_k} c_1^{j_1}\cdots c_{k-1}^{j_{k-1}} x^{j_k}.$$

$F_0 = \mathbb{Q}$. Stage $j = 1,\ldots,k_i$:

$$\begin{array}{lll} F_j = F_{j-1}, & d_j = 1 & \text{if } c_j \in F_{j-1}, \\ F_j = F_{j-1}(c_j), & d_j = \infty & \text{if } c_j \text{ is not algebraic over } F_{j-1}, \\ F_j = F_{j-1}[c_j], & d_j = m & \text{if there is an irreducible } q_j \in F_{j-1}[x], \end{array}$$

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An oracle Q with $P_{\mathbb{C}}^{Q} \neq DNP_{\mathbb{C}}^{Q}$

$$\sum_{j_1,\ldots,j_k \geq 0} \alpha_{j_1,\ldots,j_k} c_1^{j_1} \cdots c_k^{j_k} \in \{0,1\}?$$

where $k \leq k_i$, $c_1, \ldots, c_{k_i} \in \mathbb{C}$, $\alpha_{j_1,\ldots,j_k} \in \mathbb{Z}$.

$$\begin{array}{l} p(c_k) = 0? \\ p(c_k) = 1? \end{array} \quad \text{for} \quad p(x) = \sum_{j_1,\ldots,j_k \geq 0} \alpha_{j_1,\ldots,j_k} c_1^{j_1} \cdots c_{k-1}^{j_{k-1}} x^{j_k}.$$

$F_0 = \mathbb{Q}$. Stage $j = 1, \ldots, k_i$:

$$\begin{array}{lll} F_j = F_{j-1}, & d_j = 1 & \text{if } c_j \in F_{j-1}, \\ F_j = F_{j-1}(c_j), & d_j = \infty & \text{if } c_j \text{ is not algebraic over } F_{j-1}, \\ F_j = F_{j-1}[c_j], & d_j = m & \text{if there is an irreducible } q_j \in F_{j-1}[x], \\ & & \text{degree}(q_j) \geq 2 \ \&\ q_j(c_j) = 0. \end{array}$$

$$\begin{array}{lll} d_k = 1 & \Rightarrow & c_k \text{ is not necessary.} \\ d_k = \infty & \Rightarrow & p(c_k) \neq 0. \\ d_k > 2 & \Rightarrow & p(c_k) = 0 \text{ iff } q_k | p. \end{array}$$

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An oracle Q with $P_{\mathbb{C}}^{Q} \neq DNP_{\mathbb{C}}^{Q}$

The answer to $p(c_k) = 0$? and $p(c_k) = 1$? is only dependent on some

$$\mathrm{char}(c_1, \ldots, c_{k_i}) = (d_1, \ldots, d_{k_i}, q_1, \ldots, q_{k_i})$$

where $d_k \geq 2 \Rightarrow q_k(c_k) = 0$ and q_k irreducible.

An oracle Q with $\mathrm{P}_{\mathbb{C}}^{Q} \neq \mathrm{DNP}_{\mathbb{C}}^{Q}$

The answer to $p(c_k) = 0$? and $p(c_k) = 1$? is only dependent on some

$$\mathrm{char}(c_1, \ldots, c_{k_i}) = (d_1, \ldots, d_{k_i}, q_1, \ldots, q_{k_i})$$

where $d_k \geq 2 \Rightarrow q_k(c_k) = 0$ and q_k irreducible.

$i \in \mathbb{N}^+$ the code of (p_i, P_i, t_i)

$t_i \;\; = (d_1, \ldots, d_{k_i}, q_1, \ldots, q_{k_i})$

An oracle Q with $P_{\mathbb{C}}^{Q} \neq DNP_{\mathbb{C}}^{Q}$

The answer to $p(c_k) = 0?$ and $p(c_k) = 1?$ is only dependent on some

$$\mathrm{char}(c_1, \ldots, c_{k_i}) = (d_1, \ldots, d_{k_i}, q_1, \ldots, q_{k_i})$$

where $d_k \geq 2 \Rightarrow q_k(c_k) = 0$ and q_k irreducible.

$i \in \mathbb{N}^+$ the code of (p_i, P_i, t_i)

$$t_i = (d_1, \ldots, d_{k_i}, q_1, \ldots, q_{k_i})$$

$$\begin{aligned} V_i = \{\mathbf{x} \in \{0,1\}^{n_i} \mid & \forall c_1 \cdots \forall c_{k_i} (\mathrm{char}(c_1, \ldots, c_{k_i}) = t_i \\ & \& \ \mathcal{N}_i^{W_i, c_1, \ldots, c_{k_i}} \text{ rejects } (0, \ldots, 0) \in \mathbb{C}^{n_i} \\ & \& \ \mathbf{x} \text{ is not queried by } \mathcal{N}_i^{W_i, c_1, \ldots, c_{k_i}} \text{ on } (0, \ldots, 0) \in \mathbb{C}^{n_i})\} \end{aligned}$$

$$L_2 = \{\mathbf{y} \mid (\exists i \in \mathbb{N}^+)(\mathbf{y} \in \mathbb{C}^{n_i} \ \& \ V_i \neq \emptyset)\} \in \mathrm{DNP}_{\mathbb{C}}^{Q_2} \setminus \mathrm{P}_{\mathbb{C}}^{Q_2}.$$

A summary (5)

Structure	$P \neq DNP$	$DNP \neq NP$	$P^{\mathcal{Q}} \neq DNP^{\mathcal{Q}}$	$DNP^{\mathcal{Q}} \neq NP^{\mathcal{Q}}$
$(\{0,1\}; 0,1; ; =)$	?	no	the Baker-Gill-Solovay oracle	no oracle
$(\mathbb{Z}; \mathbb{Z}; +, -; =)$	yes	yes	$\emptyset$	$\emptyset$
$(\mathbb{R}; \mathbb{R}; +, -; =)$	yes	no	$\emptyset$	the Emerson oracle
$(\mathbb{R}; \mathbb{R}; \cdot, +, -; =)$	?	yes	$\mathcal{Q}_2$	$\emptyset$
$(\mathbb{C}; \mathbb{C}; \cdot, +, -; =)$	?	?	$\mathcal{Q}_2$	$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}_+$ an oracle $\mathcal{Q}_1$ derived from the Eme. oracle

A second oracle Q with $\mathrm{P}_{\mathbb{C}}^{Q} \neq \mathrm{DNP}_{\mathbb{C}}^{Q}$

$E_0 = \mathbb{Q}$, $\tau_1, \tau_2, \ldots$ where τ_{i+1} is transcendental over $E_i =_{\mathrm{df}} E_{i-1}(\tau_i)$

$$A_n = \{(v_1, \ldots, v_{2n}) \in \{0,1\}^{2n} \mid \quad \Sigma_{i=1}^{2n} v_i = n\}.$$

$$\mathcal{Q}_3 = \cup_{n=1}^{\infty}\{(\mathrm{sgn}(|v_1|), \ldots, \mathrm{sgn}(|v_{2n}|), \Sigma_{i=1}^{2n} v_i\tau_i) \in \mathbb{R}^{2n+1} \mid (v_1, \ldots, v_{2n}) \in A_n\}$$

$$L_3 = \cup_{n=1}^{\infty}\{(0, \ldots, 0, \Sigma_{i=1}^{2n} v_i\tau_i) \in \mathbb{R}^{2n+1} \mid (v_1, \ldots, v_{2n}) \in A_n\}.$$

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A second oracle $\mathcal{Q}$ with $\mathrm{P}_{\mathbb{C}}^{\mathcal{Q}} \neq \mathrm{DNP}_{\mathbb{C}}^{\mathcal{Q}}$

$E_0 = \mathbb{Q}$, $\tau_1, \tau_2, \ldots$ where τ_{i+1} is transcendental over $E_i =_{\mathrm{df}} E_{i-1}(\tau_i)$

$$A_n = \{(v_1, \ldots, v_{2n}) \in \{0,1\}^{2n} \mid \quad \Sigma_{i=1}^{2n} v_i = n\}.$$

$$\mathcal{Q}_3 = \cup_{n=1}^{\infty}\{(\mathrm{sgn}(|v_1|), \ldots, \mathrm{sgn}(|v_{2n}|), \Sigma_{i=1}^{2n} v_i\tau_i) \in \mathbb{R}^{2n+1} \mid (v_1, \ldots, v_{2n}) \in A_n\}$$

$$L_3 = \cup_{n=1}^{\infty}\{(0, \ldots, 0, \Sigma_{i=1}^{2n} v_i\tau_i) \in \mathbb{R}^{2n+1} \mid (v_1, \ldots, v_{2n}) \in A_n\}.$$

$\Rightarrow$ Each computation path of a $\mathrm{P}^{\mathcal{Q}_3}$-machine is traversed by $(0, \ldots, 0, x)$ only if x satisfies some

$$(z_1, \ldots, z_s, p_k(x)) \notin \mathcal{Q}_3, \qquad (z_1, \ldots, z_s, p_k(x)) \in \mathcal{Q}_3.$$

A second oracle Q with $P_{\mathbb{C}}^{Q} \neq DNP_{\mathbb{C}}^{Q}$

$E_0 = \mathbb{Q}$, $\tau_1, \tau_2, \ldots$ where τ_{i+1} is transcendental over $E_i =_{\mathrm{df}} E_{i-1}(\tau_i)$

$A_n = \{(v_1, \ldots, v_{2n}) \in \{0,1\}^{2n} \mid \sum_{i=1}^{2n} v_i = n\}$.

$\mathcal{Q}_3 = \cup_{n=1}^{\infty}\{(\mathrm{sgn}(|v_1|), \ldots, \mathrm{sgn}(|v_{2n}|), \sum_{i=1}^{2n} v_i\tau_i) \in \mathbb{R}^{2n+1} \mid (v_1, \ldots, v_{2n}) \in A_n\}$

$L_3 = \cup_{n=1}^{\infty}\{(0, \ldots, 0, \sum_{i=1}^{2n} v_i\tau_i) \in \mathbb{R}^{2n+1} \mid (v_1, \ldots, v_{2n}) \in A_n\}$.

$\Rightarrow$ Each computation path of a $\mathrm{P}^{\mathcal{Q}_3}$-machine is traversed by $(0, \ldots, 0, x)$ only if x satisfies some

$$(z_1, \ldots, z_s, p_k(x)) \notin \mathcal{Q}_3, \qquad (z_1, \ldots, z_s, p_k(x)) \in \mathcal{Q}_3.$$

For any $\mathrm{P}^{\mathcal{Q}_3}$-machine there is an i_0 such that

(1) $x = \sum_{i=i_0+1}^{2n} v_i\tau_i$,

(2) $v_l \neq 0$, $v_{l+1} = \cdots = v_{2n} = 0$ $(i_0 < l \leq 2n)$,

(3) $(z_1, \ldots, z_s, p_k(x)) \in \mathcal{Q}_3$

$\Rightarrow s \geq 2n$ and $(z_{i_0+1}, \ldots, z_s) = (\mathrm{sgn}(|v_{i_0+1}|), \ldots, \mathrm{sgn}(|v_l|), 0, \ldots, 0)$.

A second oracle Q with $P_{\mathbb{C}}^{Q} \neq DNP_{\mathbb{C}}^{Q}$

$E_0 = \mathbb{Q}$, $\tau_1, \tau_2, \ldots$ where τ_{i+1} is transcendental over $E_i =_{\text{df}} E_{i-1}(\tau_i)$

$A_n = \{(v_1, \ldots, v_{2n}) \in \{0,1\}^{2n} \mid \ \Sigma_{i=1}^{2n} v_i = n\}$.

$\mathcal{Q}_3 = \cup_{n=1}^{\infty} \{(\text{sgn}(|v_1|), \ldots, \text{sgn}(|v_{2n}|), \Sigma_{i=1}^{2n} v_i \tau_i) \in \mathbb{R}^{2n+1} \mid (v_1, \ldots, v_{2n}) \in A_n\}$

$L_3 = \cup_{n=1}^{\infty} \{(0, \ldots, 0, \Sigma_{i=1}^{2n} v_i \tau_i) \in \mathbb{R}^{2n+1} \mid (v_1, \ldots, v_{2n}) \in A_n\}$.

$\Rightarrow$ Each computation path of a $\text{P}^{\mathcal{Q}_3}$-machine is traversed by $(0, \ldots, 0, x)$ only if x satisfies some

$$(z_1, \ldots, z_s, p_k(x)) \notin \mathcal{Q}_3, \qquad (z_1, \ldots, z_s, p_k(x)) \in \mathcal{Q}_3.$$

For any $\text{P}^{\mathcal{Q}_3}$-machine there is an i_0 such that

(1) $x = \Sigma_{i=i_0+1}^{2n} v_i \tau_i$,

(2) $v_l \neq 0$, $v_{l+1} = \cdots = v_{2n} = 0$ $(i_0 < l \leq 2n)$,

(3) $(z_1, \ldots, z_s, p_k(x)) \in \mathcal{Q}_3$

$\Rightarrow s \geq 2n$ and $(z_{i_0+1}, \ldots, z_s) = (\text{sgn}(|v_{i_0+1}|), \ldots, \text{sgn}(|v_l|), 0, \ldots, 0)$.

$\Rightarrow L_3 \in \text{DNP}_{\mathbb{C}}^{\mathcal{Q}_3} \setminus \text{P}_{\mathbb{C}}^{\mathcal{Q}_3}$.

A summary (6)

Structure	$P \neq DNP$	$DNP \neq NP$	$P^{\mathcal{Q}} \neq DNP^{\mathcal{Q}}$	$DNP^{\mathcal{Q}} \neq NP^{\mathcal{Q}}$
$(\{0,1\}; 0, 1; ; =)$	?	no	the Baker-Gill-Solovay oracle	no oracle
$(\mathbb{Z}; \mathbb{Z}; +, -; =)$	yes	yes	$\emptyset$	$\emptyset$
$(\mathbb{R}; \mathbb{R}; +, -; =)$	yes	no	$\emptyset$	the Emerson oracle
$(\mathbb{R}; \mathbb{R}; \cdot, +, -; =)$	?	yes	$\mathcal{Q}_2$ $\mathcal{Q}_3$	$\emptyset$
$(\mathbb{C}; \mathbb{C}; \cdot, +, -; =)$	?	?	$\mathcal{Q}_2$ $\mathcal{Q}_3$	$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}_+$ an oracle $\mathcal{Q}_1$ derived from the Eme. oracle

Two oracles O with $P_{\mathbb{C}}^{O} = DNP_{\mathbb{C}}^{O}$

The universal DN-oracle and the E-oracle:

$$V_0 = \emptyset,$$
$$W_i = \cup_{j<i} V_j,$$

$$V_i = \{(\underbrace{1, \ldots, 1}_{t\times}, \mathbf{x}, Code(\mathcal{M})) \mid \mathcal{M} \text{ is a DN}^{W_i}\text{-machine } \& \ \mathcal{M}(\mathbf{x}) \downarrow^{t}\}$$

Two oracles $\mathcal{O}$ with $\mathrm{P}_{\mathbb{C}}^{\mathcal{O}} = \mathrm{DNP}_{\mathbb{C}}^{\mathcal{O}}$

The universal DN-oracle and the E-oracle:

$$V_0 = \emptyset,$$
$$W_i = \cup_{j<i} V_j,$$

$$V_i = \{(\underbrace{1,\ldots,1}_{t\times}, \mathbf{x}, Code(\mathcal{M})) \mid \mathcal{M} \text{ is a } \mathrm{DN}^{W_i}\text{-machine } \& \ \mathcal{M}(\mathbf{x}) \downarrow^{t}\}$$

$$\mathcal{O}_1 = \cup_{i\geq 1} W_i.$$

2^t

$$\mathcal{O}_2 = \{(\underbrace{1,\ldots,1}_{t\times}, \mathbf{x}, Code(\mathcal{M})) \mid \mathcal{M} \text{ is a E-machine } \& \ \mathcal{M}(\mathbf{x}) \downarrow^{2^t}\}.$$

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Two oracles $\mathcal{O}$ with $\mathrm{P}_{\mathbb{C}}^{\mathcal{O}} = \mathrm{DNP}_{\mathbb{C}}^{\mathcal{O}}$

The universal DN-oracle and the E-oracle:

$$V_0 = \emptyset,$$
$$W_i = \cup_{j<i} V_j,$$

$$V_i = \{(\underbrace{1,\ldots,1}_{t\times}, \mathbf{x}, Code(\mathcal{M})) \mid \mathcal{M} \text{ is a DN}^{W_i}\text{-machine } \& \ \mathcal{M}(\mathbf{x})\downarrow^t\}$$

non-deterministic machine

$$\mathcal{O}_1 = \cup_{i\geq 1} W_i.$$

deterministic machine

$$\mathcal{O}_2 = \{(\underbrace{1,\ldots,1}_{t\times}, \mathbf{x}, Code(\mathcal{M})) \mid \mathcal{M} \text{ is a E-machine } \& \ \mathcal{M}(\mathbf{x})\downarrow^{2^t}\}.$$

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Two oracles $\mathcal{O}$ with $P_{\mathbb{C}}{}^{\mathcal{O}} = DNP_{\mathbb{C}}{}^{\mathcal{O}}$

The universal DN-oracle and the E-oracle:

$$V_0 = \emptyset,$$
$$W_i = \cup_{j<i} V_j,$$

$$V_i = \{(\underbrace{1,\ldots,1}_{t\times},\mathbf{x},Code(\mathcal{M})) \mid \mathcal{M} \text{ is a } \mathrm{DN}^{W_i}\text{-machine } \& \ \mathcal{M}(\mathbf{x})\downarrow^t\}$$

$$\mathcal{O}_1 = \cup_{i\geq 1} W_i.$$

$$\mathcal{O}_2 = \{(\underbrace{1,\ldots,1}_{t\times},\mathbf{x},Code(\mathcal{M})) \mid \mathcal{M} \text{ is a E-machine } \& \ \mathcal{M}(\mathbf{x})\downarrow^{2^t}\}.$$

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Two oracles O with $P_{\mathbb{C}}^{O} = DNP_{\mathbb{C}}^{O}$

The universal DN-oracle and the E-oracle:

$$V_0 = \emptyset,$$
$$W_i = \cup_{j<i} V_j,$$

$$V_i = \{(\underbrace{1,\ldots,1}_{t\times},\mathbf{x},Code(\mathcal{M})) \mid \mathcal{M} \text{ is a } \mathrm{DN}^{W_i}\text{-machine } \& \ \mathcal{M}(\mathbf{x})\downarrow^{t}\}$$

$$\mathcal{O}_1 = \cup_{i\geq 1} W_i.$$

$$\mathcal{O}_2 = \{(\underbrace{1,\ldots,1}_{t\times},\mathbf{x},Code(\mathcal{M})) \mid \mathcal{M} \text{ is a E-machine } \& \ \mathcal{M}(\mathbf{x})\downarrow^{2^t}\}.$$

$$\Rightarrow \quad \mathrm{P}_{\mathbb{C}}^{\mathcal{O}_1} = \mathrm{DNP}_{\mathbb{C}}^{\mathcal{O}_1} \quad \text{and} \quad \mathrm{P}_{\mathbb{C}}^{\mathcal{O}_2} = \mathrm{DNP}_{\mathbb{C}}^{\mathcal{O}_2}.$$

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$\mathrm{DPH}_{\mathbb{C}}$ and $\mathrm{DPAT}_{\mathbb{C}}$

The Polynomial Hierarchy $\mathrm{DPH}_{\mathbb{C}} = \cup_{k\geq 0}\, \mathrm{D}\Sigma^k_{\mathbb{C}}$

$\mathcal{P} \subseteq \mathbb{C}^\infty$

$\mathcal{P} \in \mathrm{D}\Sigma^k_{\mathbb{C}}$ $(k \geq 0)$ iff there are $\mathcal{P}_0 \in \mathrm{P}_{\mathbb{C}}$, $p_1, \ldots, p_k \in \mathbb{N}[x]$:

$$\mathbf{x} \in \mathcal{P} \cap \mathbb{C}^n \Leftrightarrow$$
$$(\exists \mathbf{y}^{(1)} \in \{0,1\}^{p_1(n)})(\forall \mathbf{y}^{(2)} \in \{0,1\}^{p_2(n)}) \cdots (Q_k \mathbf{y}^{(k)} \in \{0,1\}^{p_k(n)})$$
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$\mathrm{DPH}_{\mathbb{C}}$ and $\mathrm{DPAT}_{\mathbb{C}}$

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Two oracles $\mathcal{O}$ with $P_{\mathbb{C}}^{\mathcal{O}} = DNP_{\mathbb{C}}^{\mathcal{O}}$

$$\mathrm{PSPACE}_{\mathbb{C}}$$

$$\nearrow$$

$$\mathrm{DPH}_{\mathbb{C}} \rightarrow \mathrm{DPAT}_{\mathbb{C}} \rightarrow \mathrm{P}_{\mathbb{C}}^{\mathcal{O}_1} = \mathrm{DNP}_{\mathbb{C}}^{\mathcal{O}_1}$$

$$\searrow$$

$$\mathrm{P}_{\mathbb{C}}^{\mathrm{E}_{\mathbb{C}}} = \mathrm{P}_{\mathbb{C}}^{\mathcal{O}_2} = \mathrm{DNP}_{\mathbb{C}}^{\mathcal{O}_2} = \mathrm{EXP}_{\mathbb{C}}$$

$$\mathrm{E}_{\mathbb{C}} = \mathrm{TIME}(k^n)$$
$$\mathrm{EXP}_{\mathbb{C}} = \mathrm{TIME}(2^{n^k})$$

Relativizations of the P =? DNP Question over the Complex Numbers

Thank you for your attention!

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Greifswald.

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