Hierarchies below the Halting Problem for additive machines

Christine Gaßner
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The additive BSS machines

Computation instructions:

$l: \ Z_i := Z_j + Z_k,$

$l: \ Z_i := Z_j - Z_k,$

$l: \ Z_j := c,$

Branching instructions:

$l: \ \text{if } Z_j = 0 \text{ then goto } l_1 \text{ else goto } l_2,$

$l: \ \text{if } Z_j \geq 0 \text{ then goto } l_1 \text{ else goto } l_2,$

Copy instructions:

$l: \ Z_{I_j} := Z_{I_k},$

Index instructions:

$l: \ I_j := 1,$

$l: \ I_j := I_j + 1,$

$l: \ \text{if } I_j = I_k \text{ then goto } l_1 \text{ else goto } l_2.$

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The reduction by oracle machines

$\mathsf{M}_{\mathrm{add}}$	additive BSS machines
$\mathsf{M}^1_{\mathrm{add}}$	machines in $\mathsf{M}_{\mathrm{add}}$ using only the constants 0 and 1
$\mathsf{M}^{1,=}_{\mathrm{add}}$	machines in $\mathsf{M}^1_{\mathrm{add}}$ performing only tests $Z_j = 0$
$\mathsf{M}^1_{\mathrm{add}}(\mathcal{O})$	$\mathsf{M}^1_{\mathrm{add}}$-machines using $\mathcal{O} \subseteq \cup_{i\geq 1} \mathbb{R}^i$ in

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$\mathcal{A}, \mathcal{B} \subseteq \cup_{i\geq 1} \mathbb{R}^i$

$\mathcal{A} \preceq \mathcal{B}$	$\mathcal{A}$ is *easier than* $\mathcal{B}$, $\mathcal{A}$ is decidable by a machine in $\mathsf{M}^1_{\mathrm{add}}(\mathcal{B})$
$\mathcal{A} \precneqq \mathcal{B}$	$\mathcal{A}$ is *strictly easier than* $\mathcal{B}$, $\mathcal{B}$ cannot be decided by a machine in $\mathsf{M}^1_{\mathrm{add}}(\mathcal{A})$
$\mathcal{A} \equiv \mathcal{B}$	$\mathcal{A} \preceq \mathcal{B}$ & $\mathcal{B} \preceq \mathcal{A}$

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$\mathcal{A} \equiv \mathcal{B}$	$\mathcal{A} \preceq \mathcal{B}$ & $\mathcal{B} \preceq \mathcal{A}$

$\Rightarrow$ For the Halting Problems: $\quad \mathbb{H}^{1,=}_{\mathrm{add}} \preceq \mathbb{H}^1_{\mathrm{add}} \preceq \mathbb{H}_{\mathrm{add}}.$

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The question and a first hierarchy

[Meer and Ziegler 2008]: $\mathbb{Q} \precneqq \mathbb{H}^1_{\mathrm{add}}$?

[Gaßner 2008]: $\mathbb{Q} = \mathbb{L}_1 \precneqq \mathbb{L}_2 \precneqq \cdots \precneqq \mathbb{L} \preceq \mathbb{H}^1_{\mathrm{add}}$

$$\mathbb{L} = \cup_{n\geq 1} \mathbb{L}_n$$

$$\mathbb{L}_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid (\exists (q_0, \ldots, q_{n-1}) \in \mathbb{Q}^n)(q_0 + \Sigma_{i=1}^{n-1} q_i x_i = x_n)\}$$

We have

$$\mathbb{Q} = \mathbb{L}_1 \precneqq \mathbb{L}_2 \precneqq \cdots \precneqq \mathbb{L} \preceq \mathbb{H}^{1,=}_{\mathrm{add}} \preceq \mathbb{H}^1_{\mathrm{add}} \preceq \mathbb{H}_{\mathrm{add}}.$$

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The second hierarchy

$p_1 = 2, p_2 = 3, \ldots$ an enumeration of the prime numbers.

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$p_1 = 2, p_2 = 3, \ldots$ an enumeration of the prime numbers.
$\mathcal{M}_i$ recognizes $\mathbb{R} \setminus \{\sqrt{p_i}\}$:

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Input: $x \in \mathbb{R}$;

if $x \leq 0$, then halt;

for all $j \in \mathbb{N}^+$ do {

for all $r = 0, \ldots, j-1$ do {

$q := j - r$;

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Input: $x \in \mathbb{R}$;

if $x \leq 0$, then halt;

for all $j \in \mathbb{N}^+$ do {
    for all $r = 0, \ldots, j-1$ do {
        $q := j - r$;
        if $(x < \frac{r}{q}$ and $\frac{r^2}{q^2} < p_i)$ or $(x > \frac{r}{q}$ and $\frac{r^2}{q^2} > p_i)$ then halt;
    }
}

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$$\mathbb{P}_i = \{(1, x, \mathrm{code}(\mathcal{M}_i)) \in \mathbb{H}^1_{\mathrm{add}} \mid x \in \mathbb{R} \setminus \{\sqrt{p_i}\}\},$$

$$\mathbb{H}_i = \mathbb{H}^{1,=}_{\mathrm{add}} \cup \cup_{j \le i} \mathbb{P}_j.$$

$\Rightarrow$

$$\mathbb{H}^{1,=}_{\mathrm{add}} \preceq \mathbb{H}_1 \preceq \mathbb{H}_2 \preceq \cdots \preceq \mathbb{H}^1_{\mathrm{add}}.$$

$$\mathbb{R} \setminus \{\sqrt{p_i}\} \equiv \{\sqrt{p_i}\} \preceq \{(\sqrt{p_1}, \ldots, \sqrt{p_i})\} \equiv \cup_{j \le i} \mathbb{P}_j.$$

$\mathbb{H}_i$ decidable by a machine in $\mathsf{M}^1_{\mathrm{add}}(\mathcal{O})$

$\Rightarrow \mathbb{P}_1 \cup \cdots \cup \mathbb{P}_i$ decidable by some machine in $\mathsf{M}^1_{\mathrm{add}}(\mathcal{O})$,

$\Rightarrow \{(\sqrt{p_1}, \ldots, \sqrt{p_i})\}$ decidable by some machine in $\mathsf{M}^1_{\mathrm{add}}(\mathcal{O}$

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The second hierarchy

Lemma. $S \subseteq \mathbb{R}$ *decidable by* $\mathcal{M} \in \mathsf{M}^1_{\mathrm{add}}(\mathbb{H}^{1,=}_{\mathrm{add}})$.

$\Rightarrow \exists n, m \in \mathbb{N}^+$:

$\mathcal{M}$ *rejects* $\sqrt{2}$ *and* $\frac{n}{m}\pi$ *or* $\mathcal{M}$ *accepts the both inputs.*

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$\mathcal{M}$ *rejects* $\sqrt{2}$ *and* $\frac{n}{m}\pi$ *or* $\mathcal{M}$ *accepts the both inputs.*

Proof. 1. For any path P of $\mathcal{M}$ there is a finite system

$$k_\nu x + l_\nu \geq 0 \quad \text{and} \quad k_\mu x + l_\mu > 0,$$

$$(j, k_1 x + l_1, \ldots, k_j x + l_j, \mathrm{code}(\mathcal{N})) \in \mathbb{H}^{1,=}_{\mathrm{add}},$$

$$(j, k_1 x + l_1, \ldots, k_j x + l_j, \mathrm{code}(\mathcal{N})) \notin \mathbb{H}^{1,=}_{\mathrm{add}} \ (k_i, l_i \in \mathbb{Z})$$

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which is satisfied by $x \in \mathbb{R}$ iff P is traversed by x.

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Lemma. $S \subseteq \mathbb{R}$ *decidable by* $\mathcal{M} \in \mathsf{M}^1_{\mathrm{add}}(\mathbb{H}^{1,=}_{\mathrm{add}})$.

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$\mathcal{M}$ *rejects* $\sqrt{2}$ *and* $\frac{n}{m}\pi$ *or* $\mathcal{M}$ *accepts the both inputs.*

Proof. 1. For any path P of $\mathcal{M}$ there is a finite system

$$k_\nu x + l_\nu \geq 0 \quad \text{and} \quad k_\mu x + l_\mu > 0,$$
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which is satisfied by $x \in \mathbb{R}$ iff P is traversed by x.

2. $\exists n_0, m_0 \in \mathbb{N}^+$: $\frac{n_0}{m_0}\pi$ and $\sqrt{2}$ satisfy the same system.

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The second hierarchy

$\Rightarrow$

$\{\sqrt{2}\}$ is not decidable by a machine in $\mathsf{M}^1_{\mathrm{add}}(\mathbb{H}^{1,=}_{\mathrm{add}})$.

$\mathbb{H}^{1,=}_{\mathrm{add}} \precneqq \mathbb{H}^{1,=}_{\mathrm{add}} \cup \{\sqrt{p_1}\}$.

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$\mathbb{H}^{1,=}_{\mathrm{add}} \precneqq \mathbb{H}^{1,=}_{\mathrm{add}} \cup \{\sqrt{p_1}\}$.

Lemma. *For any* $i \geq 1$,
$\{\sqrt{p_{i+1}}\}$ *is not decidable by a machine in* $\mathsf{M}^1_{\mathrm{add}}(\mathbb{H}_i)$.

Proposition 1.

$$\mathbb{H}^{1,=}_{\mathrm{add}} \precneqq \mathbb{H}_1 \precneqq \mathbb{H}_2 \precneqq \cdots \precneqq \cup_{i\geq 1} \mathbb{H}_i \preceq \mathbb{H}^1_{\mathrm{add}}.$$

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The third hierarchy for i≥2

$$\mathbb{H}_{\mathrm{spec}}(\mathsf{M}^1_{\mathrm{add}}(\mathcal{O})) = \{k_{\mathcal{M}} \in \mathbb{N}^+ \mid \mathcal{M} \in \mathsf{M}^1_{\mathrm{add}}(\mathcal{O}) \ \&\ \mathcal{M} \text{ halts on } k_{\mathcal{M}}\}$$

$$k_{\mathcal{M}} = 2^{|\mathrm{code}(\mathcal{M})|} + c_{\mathcal{M}} \text{ and } \mathrm{bin}(c_{\mathcal{M}}) = \mathrm{code}(\mathcal{M}).$$

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$c_1 = 1$ and, for $i \geq 2$,

$$c_i = \Sigma_{j=1}^{\infty} \alpha_j 10^{-j}, \quad \alpha_j = \begin{cases} 1 & \text{if } j \in \mathbb{H}_{\mathrm{spec}}(\mathsf{M}^1_{\mathrm{add}}(\mathbb{H}_{\mathrm{add}}^{c_1,\ldots,c_{i-1}})) \\ 0 & \text{otherwise,} \end{cases}$$

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➡ $\mathbb{H}^{(i)}_{\text{add}} = \cup_{n\geq 1}\{(n, \boldsymbol{x}, \text{code}^{(i)}(\mathcal{M})) \mid (n, \boldsymbol{x}, \text{code}(\mathcal{M})) \in \mathbb{H}_{\text{add}}\}.$

➡ $\text{code}^{(i)}(\mathcal{M})$ starts with i

$c_1, c_2, \ldots, c_i$ are encoded by $1, 2, \ldots$

constants in $\mathbb{R} \setminus \{c_1, c_2, \ldots, c_i\}$ by themselves

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The third hierarchy for i≥2

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$$\mathbb{H}^{c_1,\ldots,c_i}_{\text{add}} = \cup_{n\geq 1}\{(n, \boldsymbol{x}, \text{code}^{(i)}(\mathcal{M})) \in \mathbb{H}^{(i)}_{\text{add}} \mid \mathcal{M} \in \mathsf{M}^{c_1,\ldots,c_i}_{\text{add}}\}.$$

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$\mathsf{M}^{c_1,\ldots,c_i}_{\text{add}}$ machines using only the constants $c_1, \ldots, c_i$

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$\text{code}^{(i)}(\mathcal{M})$ starts with i

$c_1, c_2, \ldots, c_i$ are encoded by $1, 2, \ldots$

constants in $\mathbb{R} \setminus \{c_1, c_2, \ldots, c_i\}$ by themselves

$\mathsf{M}^{c_1,\ldots,c_i}_{\text{add}}$ machines using only the constants $c_1, \ldots, c_i$

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$\mathbb{H}_{\rm spec}(\mathsf{M}^1_{\rm add}(\mathcal{O}))$ is not decidable within $\mathsf{M}^1_{\rm add}(\mathcal{O})$.

$\mathbb{H}_{\rm spec}(\mathsf{M}^1_{\rm add}(\mathbb{H}^{c_1,\ldots,c_{i-1}}_{\rm add}))$ is decidable within $\mathsf{M}^1_{\rm add}(\mathbb{H}^{c_1,\ldots,c_i}_{\rm add})$.

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$\mathcal{N}_i$: Input: $x \in \mathbb{R}$; $c := c_i$;

for all $j = 1, 2, \ldots$ do {

$c := 10 \cdot c$;

if $j = x$ and $c > 1$ then halt;

if $c > 1$ then $c := c - 1$;

}.

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if $c > 1$ then $c := c - 1$;

}.

$\Rightarrow \{(1, x, \text{code}(\mathcal{N}_i)) \mid \mathcal{N}_i \text{ halts on } x\} \subseteq \mathbb{H}^{c_1,\ldots,c_i}_{\text{add}}$

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The third hierarchy for i≥2

Proposition 2. $\mathbb{H}^1_{\text{add}} \precnsim \mathbb{H}^{c_1,\ldots,c_i}_{\text{add}} \precnsim \mathbb{H}^{(i)}_{\text{add}} \equiv^{c_1,\ldots,c_i}_{\text{add}} \mathbb{H}_{\text{add}},\ i \geq 1.$

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Proposition 2. $\mathbb{H}^{1}_{\mathrm{add}} \precneqq \mathbb{H}^{c_1,\ldots,c_i}_{\mathrm{add}} \precneqq \mathbb{H}^{(i)}_{\mathrm{add}} \equiv^{c_1,\ldots,c_i}_{\mathrm{add}} \mathbb{H}_{\mathrm{add}},\ i \geq 1.$

$$\mathbb{H}^{c_1}_{\mathrm{add}} \to \mathbb{H}^{c_1,c_2}_{\mathrm{add}} \to \mathbb{H}^{c_1,c_2,c_3}_{\mathrm{add}} \to \cdots \to \mathbb{H}^{c_1,\ldots,c_i}_{\mathrm{add}} \to \cup_{i\geq 1}\mathbb{H}^{c_1,\ldots,c_i}_{\mathrm{add}}$$

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The third hierarchy for i≥2

Proposition 2. $\mathbb{H}^{1}_{\mathrm{add}} \underset{\neq}{\preceq} \mathbb{H}^{c_1,\ldots,c_i}_{\mathrm{add}} \underset{\neq}{\preceq} \mathbb{H}^{(i)}_{\mathrm{add}} \equiv^{c_1,\ldots,c_i}_{\mathrm{add}} \mathbb{H}_{\mathrm{add}},\ i \geq 1.$

$$
\begin{array}{ccccccccccccc}
\mathbb{H}^{c_1}_{\mathrm{add}} & \rightarrow & \mathbb{H}^{c_1,c_2}_{\mathrm{add}} & \rightarrow & \mathbb{H}^{c_1,c_2,c_3}_{\mathrm{add}} & \rightarrow & \cdots & \rightarrow & \mathbb{H}^{c_1,\ldots,c_i}_{\mathrm{add}} & \rightarrow & \cup_{i\geq 1}\mathbb{H}^{c_1,\ldots,c_i}_{\mathrm{add}} \\
\searrow & & \searrow & & \searrow & & & & \searrow & & \\
\mathbb{H}_{\mathrm{add}} & \underset{c_2}{\longleftrightarrow} & \mathbb{H}^{(2)}_{\mathrm{add}} & \underset{c_3}{\longleftrightarrow} & \mathbb{H}^{(3)}_{\mathrm{add}} & \underset{c_4}{\longleftrightarrow} & \cdots & \underset{c_i}{\longleftrightarrow} & \mathbb{H}^{(i)}_{\mathrm{add}} & &
\end{array}
$$

$\preceq^{k_1,\ldots,k_j}_{\mathrm{add}}$, $\underset{k_1,\ldots,k_j}{\rightarrow}$ reductions by using the constants $k_1,\ldots,k_j$

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$$\begin{array}{ccccccccccc}
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\searrow & & \searrow & & \searrow & & & & \searrow & & \\
\mathbb{H}_{\mathrm{add}} & \underset{c_2}{\longleftrightarrow} & \mathbb{H}^{(2)}_{\mathrm{add}} & \underset{c_3}{\longleftrightarrow} & \mathbb{H}^{(3)}_{\mathrm{add}} & \underset{c_4}{\longleftrightarrow} & \cdots & \underset{c_i}{\longleftrightarrow} & \mathbb{H}^{(i)}_{\mathrm{add}} & & \\
\updownarrow\ \mathrm{add} & & & & & & & & & & \\
\mathbb{H}^{\mathrm{ND}}_{\mathrm{add}} & & & & & & & & & &
\end{array}$$

$\mathbb{H}_{\mathrm{add}},\ \mathbb{H}^{\mathrm{ND}}_{\mathrm{add}}$	Halting Problems for the additive machines constants are encoded by themselves
$\preceq^{k_1,\ldots,k_j}_{\mathrm{add}},\ \overrightarrow{{}_{k_1,\ldots,k_j}}$	reductions by using the constants $k_1,\ldots,k_j$
$\preceq_{\mathrm{add}},\ \overrightarrow{{}_{\mathrm{add}}}$	reductions by any additive machines

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Thank you for your attention!

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