BSS RAM's with \nu-Oracle and the Axiom of Choice

Christine Gaßner

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$\Downarrow$ Stephen C. Kleene
Recursion Theory based on recursion and μ-operator

$\Downarrow$ Yiannis N. Moschovakis
Generalized Recursion Theory based on recursion and ν-operator

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ν-Operators for BSS RAM’s over arbitrary mathematical structures

BSS RAM's with \nu-Oracle and the Axiom of Choice
(History and Outline)

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Computable multi-valued correspondences

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Question:

Are there computable choice functions for these correspondences?

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Question:
Are there computable choice functions for these correspondences?
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Outline:

- BSS RAM's
- A characterization of [non-]deterministic semi-decidability
- $AC^{n,m}$ (in HPL) and effective $AC^{n,m}$ and AC^{∞}

Computation over $\mathcal{A} = (\underbrace{U_{\mathcal{A}}}_{\text{universe}} ; \underbrace{C_{\mathcal{A}}}_{\text{constants}} ; \underbrace{f_1, \ldots, f_{n_1}}_{\text{operations}} ; \underbrace{R_1, \ldots, R_{n_2}, =}_{\text{relations}})$.

Z_1	Z_2	Z_3	Z_4	Z_5	$\ldots$

Registers for elements in $U_{\mathcal{A}}$

I_1	I_2	I_3	I_4	$\ldots$	$I_{k_{\mathcal{M}}}$

Registers for indices in $\mathbb{N}$

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Registers for indices in $\mathbb{N}$

- Computation instructions:

  $\ell\colon\ Z_j := f_k(Z_{j_1}, \ldots, Z_{j_{m_k}})$ (e.g. $\ell\colon\ Z_j := Z_{j_1} + Z_{j_2}$)

  $\ell\colon\ Z_j := d_k$ $(d_k \in C_{\mathcal{A}} \subseteq U_{\mathcal{A}})$

- Branching instructions:

  $\ell\colon$ if $Z_i = Z_j$ then goto ℓ_1 else goto ℓ_2

  $\ell\colon$ if $R_k(Z_{j_1}, \ldots, Z_{j_{n_k}})$ then goto ℓ_1 else goto ℓ_2

- Copy instructions:

  $\ell\colon\ Z_{I_j} := Z_{I_k}$

- Index instructions:

  $\ell\colon\ I_j := 1$

  $\ell\colon\ I_j := I_j + 1$

  $\ell\colon$ if $I_j = I_k$ then goto ℓ_1 else goto ℓ_2

- $U_{\mathcal{A}}$ is the universe of $\mathcal{A}$
- Input and output space: $U_{\mathcal{A}}^{\infty} =_{\mathrm{df}} \bigcup_{i\geq 1} U_{\mathcal{A}}^{i}$

- Input of $\vec{x} = (x_1, \ldots, x_n) \in U_{\mathcal{A}}^{\infty}$:

x_1 x_2 x_3 x_4 x_n

Uniform Computation over Algebraic Structures
(Input and Output Procedures of Machines in $M_{\mathcal{A}}$)

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$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$		$\downarrow$	$\downarrow$	$\downarrow$	
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Z_1	Z_2	Z_3	Z_4	$\ldots$	Z_n	Z_{n+1}	Z_{n+2}	$\ldots$

I_1	I_2	I_3	I_4	$\ldots$	$I_{k_{\mathcal{M}}}$
$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$		$\uparrow$
n	1	1	1		1

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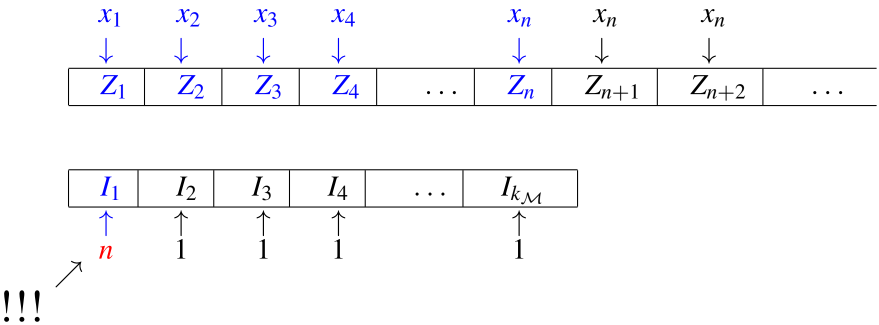

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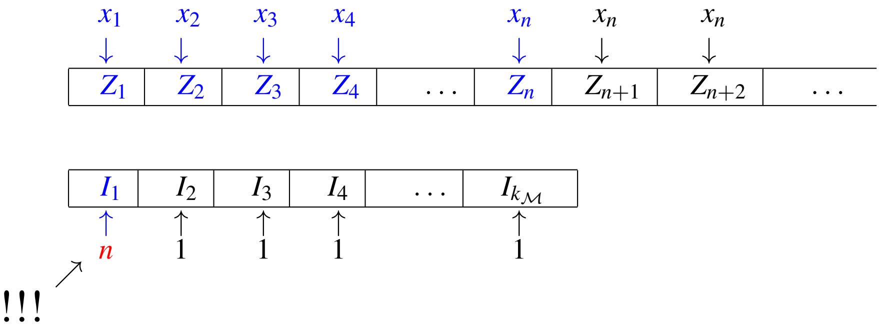

- Output of $Z_1, \ldots, Z_{I_1}$.

[ν-]Semi-Decidability
(The Definitions)

$P \subseteq U_{\mathcal{A}}^{\infty}$ is a *decision problem*.

$P \subseteq U_{\mathcal{A}}^{\infty}$ is *semi-decidable*
if there is a BSS RAM $\mathcal{M}$ such that $\vec{x} \in P \Leftrightarrow \underbrace{\mathcal{M}(\vec{x}) \text{ halts on } \vec{x}}_{\mathcal{M}(\vec{x})\downarrow}$.

We will also use:

$P \subseteq U_{\mathcal{A}}^{\infty}$ is *nondeterministically semi-decidable*
if there is a nondeterministic BSS RAM $\mathcal{M}$ such that $\vec{x} \in P$
$\Leftrightarrow \underbrace{\mathcal{M} \text{ halts on } \vec{x} \text{ for some guesses}}_{\mathcal{M}(\vec{x})\downarrow}$.

$P \subseteq U_{\mathcal{A}}^{\infty}$ is *ν-semi-decidable*
if there is a ν-oracle BSS RAM semi-deciding P.

...

ν-oracle BSS RAM = BSS RAM being able to use operator ν

...

$\mathcal{A}$ fixed, $\mathbb{N} \subseteq U_{\mathcal{A}}$ effectively enumerable over $\mathcal{A}$,
$f : U_{\mathcal{A}}^{\infty} \to \underbrace{\{a, b\}}_{\{1,0\}}$ partial function, computable over $\mathcal{A}$.

Definition (Kleene's operator for $\mathcal{A}$)

$\mu[f](x_1, \ldots, x_n)$
$=_{\text{df}} \min\{k \in \mathbb{N} \mid f(x_1, \ldots, x_n, k) = 1 \;\&\; f(x_1, \ldots, x_n, l)\downarrow \text{ for } l < k, l \in \mathbb{N}\}$

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Example

$\mathcal{A} = (\mathbb{N}; 0; +, -; \leq, =)$

$$f_0(a_1, \ldots, a_n, x) := \begin{cases} 1 & \text{if } \underbrace{x^n + a_n x^{n-1} + \cdots + a_1 x^0}_{p(x)} = 0, \\ 0 & \text{otherwise.} \end{cases}$$

$\Rightarrow \mu[f_0](a_1, \ldots, a_n) =$ the smallest zero of p

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μ-Oracle BSS RAM's with μ-Operators for $\mathbb{N} \subseteq U_{\mathcal{A}}$

(Kleene's Operator μ)

$\mathcal{A}$ fixed, $\mathbb{N} \subseteq U_{\mathcal{A}}$ effectively enumerable over $\mathcal{A}$,

$f : U_{\mathcal{A}}^{\infty} \to \underbrace{\{a, b\}}_{\{1,0\}}$ partial function, computable over $\mathcal{A}$.

Definition (Kleene's operator for $\mathcal{A}$)

$\mu[f](x_1, \ldots, x_n)$
$=_{\mathrm{df}} \min\{k \in \mathbb{N} \mid f(x_1, \ldots, x_n, k) = 1 \ \& \ f(x_1, \ldots, x_n, l)\downarrow \text{ for } l < k, l \in \mathbb{N}\}$

Definition (Oracle Instruction with Kleene's operator)

$$\ell : \ Z_j := \mu[f](\overset{z_1}{\overset{\downarrow}{Z_1}}, \ldots, \overset{z_n}{\overset{\downarrow}{Z_{I_1}}}), \qquad \text{if } I_1 = n$$

no minimum $\Rightarrow$ the machine loops forever

Properties

Any μ-semi-decidable problem is semi-decidable over $\mathcal{A}$.

ν-Oracle BSS RAM's for Structures with a and b

(Moschovakis' Operator ν)

$\mathcal{A}$ is fixed. a, b are constants of $\mathcal{A}$.

$f : U_{\mathcal{A}}^{\infty} \to \{a, b\}$ partial function, computable over $\mathcal{A}$.

Definition (Moschovakis' operator for $\mathcal{A}$)

$$\nu[f](x_1, \ldots, x_n)$$
$$=_{\mathrm{df}} \{y_1 \in U_{\mathcal{A}} \mid (\exists (y_2, \ldots, y_m) \in U_{\mathcal{A}}^{\infty})(f(x_1, \ldots, x_n, \underbrace{y_1, y_2, \ldots, y_m}_{\vec{y} \in U_{\mathcal{A}}^{\infty}}) = a)\}$$

Definition (Oracle instruction with Moschovakis' operator)

NONDETERMINISTIC!

$$\begin{array}{c} z_1 \;\cdots\; z_n \\ \downarrow \qquad \downarrow \\ \ell :\ Z_j := \nu[f](Z_1, \ldots, Z_{I_1}) \end{array}$$

$\nu[f](z_1, \ldots, z_n) \neq \emptyset \quad \Rightarrow \quad Z_j$ contains some $z \in \nu[f](z_1, \ldots, z_n)$.

$\nu[f](z_1, \ldots, z_n) = \emptyset \quad \Rightarrow \quad$ no stop (the machine loops forever).

Nondeterministic Machines versus ν-oracle Machines
(Guessing Solutions and Nondeterministic Semi-Decidability)

$f : U_{\mathcal{A}}^{\infty} \to \{a, b\}$ partial function, computable by $\mathcal{M}_f$ over $\mathcal{A}$.

Properties (By ν-operator of a ν-oracle machine)

$$\underset{\substack{\downarrow \\ y_1}}{Z_j} := \nu[f](\overset{\substack{x_1 \\ \downarrow}}{Z_1}, \ldots, \overset{\substack{x_n \\ \downarrow}}{Z_{I_1}}); \; \ldots \; \underset{\substack{\downarrow \\ y_2}}{Z_j} := \nu[f](\overset{\substack{x_1 \\ \downarrow}}{Z_1}, \ldots, \overset{\substack{x_n \\ \downarrow}}{Z_{I_1-1}}, \overset{\substack{y_1 \\ \downarrow}}{Z_{I_1}}) \quad \ldots$$

$$\Rightarrow f(x_1, \ldots, x_n, y_1, \ldots, y_m) = a$$

Properties (By input-guessing procedure of nondeterm. machine)

$x_1 \downarrow$	$x_2 \downarrow$		$x_n \downarrow$	$y_1 \downarrow$	$y_2 \downarrow$		$y_m \downarrow$	$x_n \downarrow$	
Z_1	Z_2	$\ldots$	Z_n	Z_{n+1}	Z_{n+2}	$\ldots$	Z_{n+m}	Z_{n+m+1}	$\ldots$

Then, simulate $\mathcal{M}_f$.

Proposition

$A \subseteq U_{\mathcal{A}}^{\infty}$ is ν-semi-decidable iff A is nondeterm. semi-decidable.

ν_n-Oracle BSS RAM's versus ν-Oracle BSS RAM's
(For Motivation: Computable Choice Functions?)

$\mathcal{A} = (\mathbb{N}; \mathbb{N}; ; =)$.

$$R(x_1, \ldots, x_n) := \begin{cases} 1 & \text{if } x_i \neq x_j \text{ for all } i, j \text{ with } i \neq j, \\ 0 & \text{otherwise.} \end{cases}$$

Example (ν_n for fixed arity n)

$$\begin{array}{ccc} (z_1, \ldots, z_n) & \in \mathbb{N}^n \\ \downarrow \quad\quad \downarrow & \\ \ell : \; Z_j := \nu_n[R](Z_1, \ldots, Z_n) & \end{array}$$

Example (ν for any arity)

$$\begin{array}{ccc} (z_1, \ldots, z_n) & \in \mathbb{N}^\infty \\ \downarrow \quad\quad \downarrow & \\ \ell : \; Z_j := \nu[R](Z_1, \ldots, Z_{I_1}) & \end{array}$$

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Properties

In both cases, we get a $z \in \mathbb{N} \setminus \{z_1, \ldots, z_n\}$ if $z_i \neq z_{i+k}$. $(k \geq 1)$

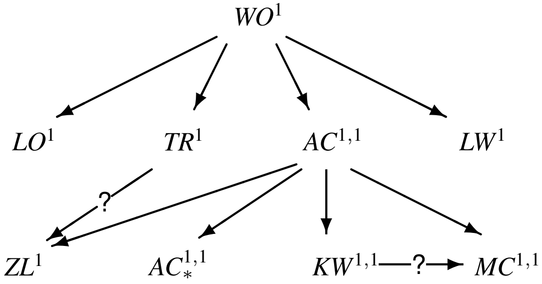
For the ν_n-computable correspondence $\mathbb{N}^n \ni (z_1, \ldots, z_n) \mapsto \mathbb{N} \setminus \{z_1, \ldots, z_n\}$ we have a choice function computable by means of $n + 1$ constants.

For the ν-computable correspondence $\mathbb{N}^\infty \ni (z_1, \ldots, z_n) \mapsto \mathbb{N} \setminus \{z_1, \ldots, z_n\}$ we do not have a computable choice function.

The Axiom of Choice in the Second-Order Logic
(Some Definitions and Relationships between Statements Related to AC in HPL)

$$
\begin{aligned}
AC^{n,m} &=_{\text{df}} \forall A \forall R \exists S(cor(R,A) \to \forall \vec{X}(A\vec{X} \to \exists!!\vec{W}(R(\vec{X}\,.\,\vec{W}) \wedge S(\vec{X}\,.\,\vec{W})))) \\
WO^{n} &=_{\text{df}} \forall A \exists T(wo(T,A)) \\
LO^{n} &=_{\text{df}} \forall A \exists T(lo(T,A)) \\
&\ \ \ldots
\end{aligned}
$$

For Henkin-structures (satisfying the axioms of comprehension):



If we find neither $H_1 -? \to H_2$ nor, for a statement H_3, $H_1 \longrightarrow H_3$ and $H_3 \longrightarrow H_2$, then $H_1 \to H_2$ is not deducible from ${}^{h}ax$. (Gaßner 1994)

Effective Second Order Logic and a Generalization
(The Axiom of Choice)

Definition (An effective form of the axiom of choice over $\mathcal{A}$)

A semi-decidable
R semi-decidable correspondence with domain A
$\Rightarrow$ There is a semi-decidable mapping S such that $(*)$ is satisfied.

Details (An effective $AC^{n,m}$)

$A \subseteq U_{\mathcal{A}}^{n}$
$R \subseteq U_{\mathcal{A}}^{n+m}$
$S \subseteq U_{\mathcal{A}}^{n+m}$

$$\overbrace{\forall \vec{X}(A\vec{X} \rightarrow \exists!!\vec{Y}(R(\vec{X}\,.\,\vec{Y}) \wedge S(\vec{X}\,.\,\vec{Y})))}^{(*)}$$

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$R(\vec{X}\,.\,\vec{Y})$ means $(X_1, \ldots, X_n, Y_1, \ldots, Y_m) \in R$ (for tuples in $U_{\mathcal{A}}^{n}$ and $U_{\mathcal{A}}^{m}$)

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$R\langle\vec{X},\vec{Y}\rangle$ means $(X_1,a,\ldots,X_{n-1},a,X_n,b,Y_1,a,\ldots,Y_{m-1},a,Y_m) \in R$ (for tuples in $U_{\mathcal{A}}^{\infty}$)

Structures with AC^∞ and without AC^∞
(Some Examples)

Example (Structures with effective AC^∞)

- $(\{0,1\}; 0, 1; ; =)$
- $(\mathbb{N}; 0; s; =), \qquad s(n) = n + 1$
- $(\mathbb{Q}; \mathbb{Q}; +, -; \leq, =)$
- $(\mathbb{R}; \mathbb{R}; +, -; \leq, =)$
- $\ldots$

Example (Structures without effective AC^∞)

- ?

Properties (Representation of $\mathcal{A}$ by predicates)

$$REL_{\mathcal{A}} =_{\text{df}} \{\underbrace{R_1, \ldots, R_{n_2}}_{R_j \subseteq U_{\mathcal{A}}^{n_j}}, \underbrace{F_1, \ldots, F_{n_1}}_{F_j = \{(x_1,\ldots,x_{m_j},y) \mid y = f_j(x_1,\ldots,x_{m_j})\} \subseteq U_{\mathcal{A}}^{m_j+1}}\}$$

π permutation of $U_{\mathcal{A}}$

$$\pi(A) =_{\text{df}} \bigcup_n \{(\pi(x_1), \ldots, \pi(x_n)) \mid (x_1, \ldots, x_n) \in A\} \qquad (A \subseteq U_{\mathcal{A}}^{\infty})$$

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$\mathcal{G}$ is the group of relations-preserving automorphisms π of $U_{\mathcal{A}}$ with

$$\pi(R_j) = R_j,$$
$$\pi(F_j) = F_j.$$

A Characterization of Semi-Decidability
(Transfer of a Method from the Second-Order Logic)

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Definition (Some subgroups of $\mathcal{G}$)

$$\mathcal{G}(P) =_{\text{df}} \{\pi \in \mathcal{G} \mid (\forall x \in P)(\pi(x) = x)\} \qquad (P \subseteq U_{\mathcal{A}})$$

$$sym_{\mathcal{G}}(A) =_{\text{df}} \{\pi \in \mathcal{G} \mid \pi(A) = A\} \qquad (A \subseteq U_{\mathcal{A}}^{\infty})$$

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Theorem (A property of [non-deterministic] semi-decidability)

For any $A \subseteq U_{\mathcal{A}}^{\infty}$ that is [non-deterministically] semi-dec. over $\mathcal{A}$, there is a finite $P \subseteq U_{\mathcal{A}}$ such that $\mathcal{G}(P) \subseteq sym_{\mathcal{G}}(A)$.

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Definition (Some subgroups of $\mathcal{G}$)

$$\mathcal{G}(P) =_{\text{df}} \{\pi \in \mathcal{G} \mid (\forall x \in P)(\pi(x) = x)\} \qquad (P \subseteq U_{\mathcal{A}})$$
$$sym_{\mathcal{G}}(A) =_{\text{df}} \{\pi \in \mathcal{G} \mid \pi(A) = A\} \qquad (A \subseteq U_{\mathcal{A}}^{\infty})$$

Theorem (A property of [non-deterministic] semi-decidability)

For any $A \subseteq U_{\mathcal{A}}^{\infty}$ that is [non-deterministically] semi-dec. over $\mathcal{A}$, there is a finite $P \subseteq U_{\mathcal{A}}$ such that $\mathcal{G}(P) \subseteq sym_{\mathcal{G}}(A)$.

More general:
$\mathcal{G}$ is the group of permutations of $U_{\mathcal{A}}$.
Let $\mathcal{I}_{\mathcal{A}} \subseteq \mathcal{P}(U_{\mathcal{A}} \cup REL_{\mathcal{A}})$ be a normal ideal in $U_{\mathcal{A}}$ with respect to $\mathcal{G}$.

Theorem (Normal ideals and [non-deterministic] semi-decidability)

For any $A \subseteq U_{\mathcal{A}}^{\infty}$ that is [non-deterministically] semi-dec. over $\mathcal{A}$, there is a $P \in \mathcal{I}_{\mathcal{A}}$ such that $\mathcal{G}(P) \subseteq sym_{\mathcal{G}}(A)$.

Example (Structures with effective AC^∞)

- $(\{0,1\};0,1;;=)$
- $(\mathbb{N};0;s;=),\quad s(n)=n+1$
- $(\mathbb{Q};\mathbb{Q};+,-;\leq,=)$
- $(\mathbb{R};\mathbb{R};+,-;\leq,=)$
- $\ldots$

Structures with AC^∞ and without AC^∞
(Some Examples)

Example (Structures with effective AC^∞)

- $(\{0,1\}; 0,1; ; =)$
- $(\mathbb{N}; 0; s; =), \quad s(n) = n+1$
- $(\mathbb{Q}; \mathbb{Q}; +, -; \leq, =)$
- $(\mathbb{R}; \mathbb{R}; +, -; \leq, =)$
- $\ldots$

Example (Structures without effective AC^∞)

- $(\mathbb{N}; \mathbb{N}; ; =)$
- $(\mathbb{N} \times \mathbb{N}; \mathbb{N} \times \{0\}; f; \leq_{\mathrm{lexi}}, =), \quad f(n,m) = (n,0)$

  (Note: $\leq_{\mathrm{lexi}}$ is a decidable well-ordering on $\mathbb{N} \times \mathbb{N}$.)
- $(\mathbb{Q}; \mathbb{N}; s; \leq, =), \quad s(n) = n+1$
- $\ldots$

Thank you very much for your attention!

References

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