Hierarchies of Decision Problems over Algebraic Structures Defined by Quantifiers

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A shortened version of the slides that were presented at the
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Subject:

- BSS RAM model over first order structures
  - a framework for study of
    - the abstract computability by machines over several structures
    - the uniform abstract decidability and the reducibility of decision problems over algebraic structures

    on a high abstraction level
  - includes several types of register machines, the Turing machine, and the uniform BSS model of computation over the reals
- hierarchies of undecidable decision problems within this model

Meaning:

- better understanding
  - the structural complexity of decision problems
  - the methods used in the recursion theory
  - the limits of computations over several structures

Outline

- The BSS RAM's
  - uniform machines over first order structures
- Halting problems
  - uniformity and codes for machines
- Known hierarchies
  - derived from the arithmetical hierarchy
  - Kleene–Mostowski, Cucker, ...
- A hierarchy over first order structures
  - defined by quantifiers
  - characterized by halting problems
  - complete problems

Computation over $\mathcal{A} = (\underbrace{U_\mathcal{A}}_{\text{universe}}; \underbrace{U_\mathcal{A}}_{\text{constants}}; \underbrace{f_1, \ldots, f_{n_1}}_{\text{operations}}; \underbrace{R_1, \ldots, R_{n_2}, =}_{\text{relations}})$.

Z_1	Z_2	Z_3	Z_4	$\ldots$

Registers for elements in $U_\mathcal{A}$

I_1	I_2	I_3	I_4	$\ldots$

Registers for indices / addresses

- Computation instructions:

  $\ell\colon\ Z_j := f_k(Z_{j_1}, \ldots, Z_{j_{m_k}})$ (e.g. $\ell\colon\ Z_j := Z_{j_1} + Z_{j_2}$)

  $\ell\colon\ Z_j := d_k$ $(d_k \in U_\mathcal{A})$

- Branching instructions:

  $\ell\colon$ if $Z_i = Z_j$ then goto ℓ_1 else goto ℓ_2

  $\ell\colon$ if $R_k(Z_{j_1}, \ldots, Z_{j_{n_k}})$ then goto ℓ_1 else goto ℓ_2

- Copy instructions:

  $\ell\colon\ Z_{I_j} := Z_{I_k}$

- Index instructions:

  $\ell\colon\ I_j := 1$

  $\ell\colon\ I_j := I_j + 1$

  $\ell\colon$ if $I_j = I_k$ then goto ℓ_1 else goto ℓ_2

- $U_{\mathcal{A}}^{\infty} =_{\mathrm{df}} \bigcup_{i \geq 1} U_{\mathcal{A}}^{i}$ — input and output space (for the universe $U_{\mathcal{A}}$)
- Input of $\vec{x} = (x_1, \ldots, x_n) \in U_{\mathcal{A}}^{\infty}$:

x_1 ↓	x_2 ↓	x_3 ↓		x_n ↓	x_n ↓	x_n ↓	
Z_1	Z_2	Z_3	$\ldots$	Z_n	Z_{n+1}	Z_{n+2}	$\ldots$

I_1	I_2	I_3	I_4	$\ldots$	$I_{k_{\mathcal{M}}}$
↑ n	↑ 1	↑ 1	↑ 1		↑ 1

$k_{\mathcal{M}}$ index registers

- Input and guessing procedures of nondeterministic machines:

x_1 ↓	x_2 ↓		x_n ↓	y_1 ↓	y_2 ↓		y_m ↓	x_n ↓	
Z_1	Z_2	$\ldots$	Z_n	Z_{n+1}	Z_{n+2}	$\ldots$	Z_{n+m}	Z_{n+m+1}	$\ldots$

I_1	I_2	I_3	I_4	$\ldots$	$I_{k_{\mathcal{M}}}$
↑ n	↑ 1	↑ 1	↑ 1		↑ 1

$m \in \mathbb{N}^{+}$ is also guessed.

- Output of $Z_1, \ldots, Z_{I_1}$.

- $\mathcal{A}$ is fixed.
- The first hierarchy defined semantically by deterministic machines:

$$\begin{aligned}
\Sigma_0^0 &= \mathrm{DEC}_{\mathcal{A}}\\
\Pi_n^0 &= \{U_{\mathcal{A}}^{\infty} \setminus P \mid P \in \Sigma_n^0\}\\
\Delta_n^0 &= \Sigma_n^0 \cap \Pi_n^0\\
\Sigma_{n+1}^0 &= \{P \subseteq U_{\mathcal{A}}^{\infty} \mid (\exists Q \in \Pi_n^0)(P \in \mathrm{SDEC}_{\mathcal{A}}^{Q})\}
\end{aligned}$$

- The second hierarchy defined syntactically by formulas:

$$\begin{aligned}
\Sigma_0^{\mathrm{ND}} &= \mathrm{DEC}_{\mathcal{A}}\\
\Pi_n^{\mathrm{ND}} &= \{U_{\mathcal{A}}^{\infty} \setminus P \mid P \in \Sigma_n^{\mathrm{ND}}\}\\
\Delta_n^{\mathrm{ND}} &= \Sigma_n^{\mathrm{ND}} \cap \Pi_n^{\mathrm{ND}}\\
\Sigma_{n+1}^{\mathrm{ND}} &= \{P \subseteq U_{\mathcal{A}}^{\infty} \mid (\exists Q \in \Pi_n^{\mathrm{ND}})\\
&\qquad \forall \vec{x}(\vec{x} \in P \Leftrightarrow (\exists \vec{y} \in U_{\mathcal{A}}^{\infty})((\vec{y}\,.\,\vec{x}) \in Q))\}
\end{aligned}$$

The Arithmetical Hierarchy (Kleene-Mostowski)

For Turing Machines

For $(\{0,1\};0,1;;=)$, both definitions provide the same hierarchy:

$$\Sigma^0_{n+1} = \{P \subseteq \{0,1\}^\infty \mid (\exists Q \in \Pi^0_n)(P \in \mathrm{SDEC}^Q)\}$$

$$\|$$

$$\Sigma^{\mathrm{ND}}_{n+1} = \{P \subseteq \{0,1\}^\infty \mid (\exists Q \in \Pi^0_n) \; \forall \vec{x}(\vec{x} \in P \Leftrightarrow (\exists \vec{y} \in \{0,1\}^\infty)((\vec{y} \,.\, \vec{x}) \in Q))\}$$

$$\begin{array}{ccccccccccc}
 & & \Sigma^0_1 & & & & \Sigma^0_2 & & & & \Sigma^0_3 \quad \cdots \\
 & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\
\Delta^0_1 & & & & \Delta^0_2 & & & & \Delta^0_3 & & \\
 & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & \\
 & & \Pi^0_1 & & & & \Pi^0_2 & & & & \Pi^0_3 \quad \cdots
\end{array}$$

"$\rightarrow$" means "strong $\subset$".

Complete Problems in the Arithmetical Hierarchy

For Turing Machines

$$
\begin{array}{ccccccccc}
 & & \vdots & & & & \vdots & & \\
\text{CoFIN} & \in & \Sigma^0_3 & & & & \Pi^0_3 & & \\
 & & & \nwarrow & & \nearrow & & & \\
 & & & & \Delta^0_3 & & & & \\
 & & & \nearrow & & \nwarrow & & & \\
\text{FIN} & \in & \Sigma^0_2 & & & & \Pi^0_2 & \ni & \text{TOTAL} \\
 & & & \nwarrow & & \nearrow & & & \\
 & & & & \Delta^0_2 & & & & \\
 & & & \nearrow & & \nwarrow & & & \\
\text{H}^{\text{spec}} & \in & \Sigma^0_1 & & & & \Pi^0_1 & \ni & \text{EMPTY} \\
 & & & \nwarrow & & \nearrow & & & \\
 & & & & \Delta^0_1 & & & &
\end{array}
$$

$$
\begin{aligned}
\text{CoFIN} &= \{\text{code}(\mathcal{M}) \mid (\exists n \in \mathbb{N})(\forall \vec{x} \in \{0,1\}^{(\geq n)})(\mathcal{M}(\vec{x}) \downarrow)\} \\
\text{FIN} &= \{\text{code}(\mathcal{M}) \mid (\exists n \in \mathbb{N})(\forall \vec{x} \in \{0,1\}^{(\geq n)})(\mathcal{M}(\vec{x}) \uparrow)\} \\
\text{TOTAL} &= \{\text{code}(\mathcal{M}) \mid (\forall \vec{x} \in \{0,1\}^{\infty})(\mathcal{M}(\vec{x}) \downarrow)\} \\
\text{H}^{\text{spec}} &= \{\text{code}(\mathcal{M}) \mid \mathcal{M}(\text{code}(\mathcal{M})) \downarrow\} \\
\text{EMPTY} &= \{\text{code}(\mathcal{M}) \mid (\forall \vec{x} \in \{0,1\}^{\infty})(\mathcal{M}(\vec{x}) \uparrow)\}
\end{aligned}
$$

(cp. Soare, Kozen)

Complete Problems in the BSS model (Cucker)

For Computation over the Ring of Reals $\mathbb{R} = (\mathbb{R}; \mathbb{R}; \cdot, +, -; \leq)$

$$
\begin{array}{ccccccccc}
 & & & & \vdots & & & & \\
 & & \Sigma_3^{\mathrm{ND}} & \nearrow\nwarrow & & \nwarrow\nearrow & \Pi_3^{\mathrm{ND}} & & \\
 & & & & \Delta_3^{\mathrm{ND}} & & & & \\
\text{Suslin's proj. hier.} & = & \Sigma_2^{\mathrm{ND}} & \nearrow\nwarrow & & \nwarrow\nearrow & \Pi_2^{\mathrm{ND}} & \ni & \color{blue}{\mathrm{TOTAL}_{\mathbb{R}}, \mathrm{TOTAL}_{\mathbb{R}}^{\mathrm{ND}}} \\
\text{set of Borel sets} & = & & & \Delta_2^{\mathrm{ND}} & & & & \\
(\subseteq \mathbb{R}^{\infty}) & & & & \uparrow & & & & \\
 & & & \vdots & & \vdots & & & \\
 & & & \nwarrow & & \nearrow & & & \\
 & & & & \vdots & & & & \\
\color{blue}{\mathrm{FIN}_{\mathbb{R}}} & \in & \Sigma_2^0 & \nearrow\nwarrow & & \nwarrow\nearrow & \Pi_2^0 & & \\
 & & & & \Delta_2^0 & & & & \\
\color{red}{\Sigma_1^{\mathrm{ND}}} & \color{red}{=} & \color{red}{\Sigma_1^0} & \nearrow\nwarrow & & \nwarrow\nearrow & \color{red}{\Pi_1^0} & \color{red}{=} & \color{red}{\Pi_1^{\mathrm{ND}}} \ni \color{blue}{\mathrm{INJ}_{\mathbb{R}}} \\
 & & & & \Delta_1^0 & & & &
\end{array}
$$

$$
\begin{aligned}
\mathrm{FIN}_{\mathbb{R}} &= \{\mathrm{code}(\mathcal{M}) \mid (\exists n \in \mathbb{N})(\forall \vec{x} \in \mathbb{R}^{(\geq n)})(\mathcal{M}(\vec{x}) \uparrow)\} \\
\mathrm{INJ}_{\mathbb{R}} &= \{\mathrm{code}(\mathcal{M}) \mid (\forall \vec{x}_1, \vec{x}_2 \in \mathbb{R}^{\infty})(\mathcal{M}(\vec{x}_1)\downarrow = \mathcal{M}(\vec{x}_2)\downarrow \Rightarrow \vec{x}_1 = \vec{x}_2)\} \\
\mathrm{TOTAL}_{\mathbb{R}}^{[\mathrm{ND}]} &= \{\mathrm{code}(\mathcal{M}) \mid \mathcal{M} \in \mathsf{M}_{\mathbb{R}}^{[\mathrm{ND}]} \ \&\ (\forall \vec{x} \in \mathbb{R}^{\infty})(\mathcal{M}(\vec{x}) \downarrow)\}
\end{aligned}
$$

(cp. Cucker)

For $\mathcal{A}$:

- a finite number of operations & relations, all elements are constants,
- contains an infinite set effectively enumerable over $\mathcal{A}$: $\mathbb{N} \subseteq U_{\mathcal{A}}$.

Recall (the definition):

$$\begin{aligned} \Sigma_0^0 &= \mathrm{DEC}_{\mathcal{A}} \\ \Pi_n^0 &= \{U_{\mathcal{A}}^{\infty} \setminus P \mid P \in \Sigma_n^0\} \\ \Delta_n^0 &= \Sigma_n^0 \cap \Pi_n^0 \\ \Sigma_{n+1}^0 &= \{P \subseteq U_{\mathcal{A}}^{\infty} \mid (\exists Q \in \Pi_n^0)(P \in \mathrm{SDEC}_{\mathcal{A}}^{Q})\} \end{aligned}$$

$$\Rightarrow \Sigma_{n+1}^0 = \mathrm{SDEC}_{\mathcal{A}}^{\mathbb{H}_{\mathcal{A}}^{(n)}} = \{P \subseteq U_{\mathcal{A}}^{\infty} \mid P \preceq_1 \mathbb{H}_{\mathcal{A}}^{(n+1)}\}$$

Proposition (G. 2014)

$$\Sigma_{n+1}^0 = \{P \subseteq U_{\mathcal{A}}^{\infty} \mid (\exists Q \in \Pi_n^0)\forall \vec{x}(\vec{x} \in P \Leftrightarrow (\exists k \in \mathbb{N})((\vec{x}\,.\,k) \in Q))\}$$

Note: $\mathbb{H}_{\mathcal{A}}^{(0)} = \emptyset$

$\mathbb{H}_{\mathcal{A}}^{(n+1)} =$ Halting problem for BSS RAM's using $\mathbb{H}_{\mathcal{A}}^{(n)}$ as oracle

$\mathcal{A}$: a finite number of operations & relations, all elements $\hat{=}$ constants.

Recall (the definition):

$$\begin{aligned}
\Sigma_0^{\mathrm{ND}} &= \mathrm{DEC}_{\mathcal{A}} \\
\Pi_n^{\mathrm{ND}} &= \{U_{\mathcal{A}}^{\infty} \setminus P \mid P \in \Sigma_n^{\mathrm{ND}}\} \\
\Delta_n^{\mathrm{ND}} &= \Sigma_n^{\mathrm{ND}} \cap \Pi_n^{\mathrm{ND}} \\
\Sigma_{n+1}^{\mathrm{ND}} &= \{P \subseteq U_{\mathcal{A}}^{\infty} \mid (\exists Q \in \Pi_n^{\mathrm{ND}}) \\
&\qquad \forall \vec{x}(\vec{x} \in P \Leftrightarrow (\exists \vec{y} \in U_{\mathcal{A}}^{\infty})((\vec{y}\,.\,\vec{x}) \in Q))\}
\end{aligned}$$

Proposition (G. 2015)

$$\Sigma_{n+1}^{\mathrm{ND}} = \{P \subseteq U_{\mathcal{A}}^{\infty} \mid (\exists Q \in \Pi_n^{\mathrm{ND}})(P \in (\mathrm{SDEC}_{\mathcal{A}}^{\mathrm{ND}})^{Q})\}$$

$$\Rightarrow \quad \Sigma_{n+1}^{\mathrm{ND}} = (\mathrm{SDEC}_{\mathcal{A}}^{\mathrm{ND}})^{(\mathbb{H}_{\mathcal{A}}^{\mathrm{ND}})^{(n)}} = \{P \subseteq U_{\mathcal{A}}^{\infty} \mid P \preceq_1 (\mathbb{H}_{\mathcal{A}}^{\mathrm{ND}})^{(n+1)}\}$$

Note: $(\mathbb{H}_{\mathcal{A}}^{\mathrm{ND}})^{(0)} = \emptyset$

$(\mathbb{H}_{\mathcal{A}}^{\mathrm{ND}})^{(n+1)}$ = Halting p. for ND-machines using $(\mathbb{H}_{\mathcal{A}}^{\mathrm{ND}})^{(n)}$ as oracle

Complete Problems in the First Hierarchy

For BSS RAM's — Computation over Several Structures

For $\mathcal{A}$:

- a finite number of operations & relations, all elements are constants,
- contains an infinite set effectively enumerable over $\mathcal{A}$: $\mathbb{N} \subseteq U_{\mathcal{A}}$.

$$
\begin{array}{ccccccccc}
 & & & & \vdots & & & & \\
 & & & \nearrow & & \nwarrow & & & \\
\mathrm{FIN}_{\mathbb{N}} & \in & \Sigma_2^0 & & & & \Pi_2^0 & \ni & \mathrm{TOTAL}_{\mathbb{N}}, \mathrm{INCL}_{\mathbb{N}} \\
 & & & \nwarrow & & \nearrow & & & \\
 & & & & \Delta_2^0 & & & & \\
 & & & \nearrow & & \nwarrow & & & \\
\mathbb{H}_{\mathcal{A}}^{\mathrm{spec}}, \mathbb{H}_{\mathcal{A}} & \in & \Sigma_1^0 & & & & \Pi_1^0 & & \\
 & & & \nwarrow & & \nearrow & & & \\
 & & & & \Delta_1^0 & & & &
\end{array}
$$

$$
\begin{array}{lcll}
\mathrm{FIN}_{\mathbb{N}} & = & \{\mathrm{code}(\mathcal{M}) \in U_{\mathcal{A}}^{\infty} \mid |H_{\mathcal{M}} \cap \mathbb{N}^{\infty}| < \infty\} & (H_{\mathcal{M}} = \text{halting set}) \\
\mathrm{TOTAL}_{\mathbb{N}} & = & \{\mathrm{code}(\mathcal{M}) \in U_{\mathcal{A}}^{\infty} \mid (\forall \vec{x} \in \mathbb{N}^{\infty})(\mathcal{M}(\vec{x}) \downarrow)\} & \\
\mathrm{INCL}_{\mathbb{N}} & = & \{(\mathrm{code}(\mathcal{M}) \,.\, \mathrm{code}(\mathcal{N})) \in U_{\mathcal{A}}^{\infty} \mid (H_{\mathcal{M}} \cap \mathbb{N}^{\infty}) \subseteq (H_{\mathcal{N}} \cap \mathbb{N}^{\infty})\} & \\
\mathbb{H}_{\mathcal{A}}^{[\mathrm{spec}]} & \hat{=} & \text{Halting problems for BSS RAM's over } \mathcal{A} & (\text{cp. Gaßner})
\end{array}
$$

Complete Problems in the Second Hierarchy (Blue)

$\mathcal{A}$: a finite number of operations & relations, all elements $\hat{=}$ constants.

$$\vdots$$

$$\text{TOTFIN}_{\mathcal{A}}^{\text{ND}} \in \Sigma_3^{\text{ND}} \quad \nearrow\nwarrow \quad \nwarrow\nearrow \quad \Pi_3^{\text{ND}}$$

$$\Delta_3^{\text{ND}}$$

$$\text{FIN}_{\mathcal{A}}^{\text{ND}} \in \Sigma_2^{\text{ND}} \quad \nearrow\nwarrow \quad \nwarrow\nearrow \quad \Pi_2^{\text{ND}} \ni \text{TOTAL}_{\mathcal{A}}^{\text{ND}}, \text{INCL}_{\mathcal{A},i}^{\text{ND}}, \text{CONST}_{\mathcal{A}}^{\text{ND}}$$

$$\Delta_2^{\text{ND}}$$

$$\mathbb{H}_{\mathcal{A}}^{\text{ND}} \in \Sigma_1^{\text{ND}} \quad \nearrow\nwarrow \quad \nwarrow\nearrow \quad \Pi_1^{\text{ND}} \ni \text{INJ}_{\mathcal{A}}^{\text{ND}}$$

$$\Delta_1^{\text{ND}}$$

$$\text{TOTAL}_{\mathcal{A}}^{\text{ND}} = \{\text{code}(\mathcal{M}) \mid (\forall \vec{x} \in \mathbb{R}^{\infty})(\mathcal{M}(\vec{x}) \downarrow)\} \qquad (\mathcal{M} \in \mathsf{M}_{\mathcal{A}}^{\text{ND}})$$

$$\text{INJ}_{\mathcal{A}}^{\text{ND}} = \{\text{code}(\mathcal{M}) \mid \mathcal{M} \text{ computes a/an [super] injective function}\}$$

$$\text{CONST}_{\mathcal{A}}^{\text{ND}} = \{\text{code}(\mathcal{M}) \mid \mathcal{M} \text{ computes a total constant function}\}$$

$$\text{FIN}_{\mathcal{A}}^{\text{ND}} = \{\text{code}(\mathcal{M}) \mid (\forall i \in \mathbb{N} \setminus I)(H_{\mathcal{M}} \cap U_{\mathcal{A}}^{i} = \emptyset) \text{ for some } |I| < \omega\}$$

$$\text{TOTFIN}_{\mathcal{A}}^{\text{ND}} = \{\text{code}(\mathcal{M}) \mid (\forall i \in \mathbb{N} \setminus I)(H_{\mathcal{M}} \cap U_{\mathcal{A}}^{i} \neq U_{\mathcal{A}}^{i}) \text{ for some } |I| < \omega\}$$

$$\mathbb{H}_{\mathcal{A}}^{\text{ND}} = \text{Halting problem for ND-machines over A (in } \mathsf{M}_{\mathcal{A}}^{\text{ND}})$$

$$\text{INCL}_{\mathcal{A},i}^{\text{ND}} = \{(\text{code}(\mathcal{M}) \,.\, \text{code}(\mathcal{N})) \mid (\mathcal{M},\mathcal{N}) \in \mathsf{M}_{\mathcal{A},i} \times \mathsf{M}_{\mathcal{A}}^{\text{ND}} \;\&\; H_{\mathcal{M}} \subseteq H_{\mathcal{N}}\}$$

$\mathsf{M}_{\mathcal{A},1} = \mathsf{M}_{\mathcal{A}}$, $\mathsf{M}_{\mathcal{A},2} = \mathsf{M}_{\mathcal{A}}(\mathbb{H}_{\mathcal{A}})$, $\mathsf{M}_{\mathcal{A},3} = \mathsf{M}_{\mathcal{A}}(\mathbb{H}_{\mathcal{A}}^{\text{ND}})$, $\mathsf{M}_{\mathcal{A},4} = \mathsf{M}_{\mathcal{A}}^{\text{ND}}(\mathbb{H}_{\mathcal{A}}^{\text{ND}})$

1st hierarchy:

$$\vdots$$

$$\mathrm{FIN}_{\mathbb{N}} \in \Sigma^0_2 \quad \nearrow \quad \nwarrow \quad \Pi^0_2 \ni \mathrm{TOTAL}_{\mathbb{N}}, \mathrm{INCL}_{\mathbb{N}}$$

$$\nwarrow \quad \Delta^0_2 \quad \nearrow$$

$$\mathbb{H}_{\mathcal{A}} \in \Sigma^0_1 \quad \nearrow \quad \nwarrow \quad \Pi^0_1$$

$$\nwarrow \quad \Delta^0_1 \quad \nearrow$$

2nd hierarchy:

$$\vdots$$

$$\mathrm{TOTFIN}^{\mathrm{ND}}_{\mathcal{A}} \in \Sigma^{\mathrm{ND}}_3 \quad \nearrow \quad \nwarrow \quad \Pi^{\mathrm{ND}}_3$$

$$\nwarrow \quad \Delta^{\mathrm{ND}}_3 \quad \nearrow$$

$$\mathrm{FIN}^{\mathrm{ND}}_{\mathcal{A}} \in \Sigma^{\mathrm{ND}}_2 \quad \nearrow \quad \nwarrow \quad \Pi^{\mathrm{ND}}_2 \ni \mathrm{TOTAL}^{\mathrm{ND}}_{\mathcal{A}}, \mathrm{INCL}^{\mathrm{ND}}_{\mathcal{A},i}, \mathrm{CONST}^{\mathrm{ND}}_{\mathcal{A}}$$

$$\nwarrow \quad \Delta^{\mathrm{ND}}_2 \quad \nearrow$$

$$\mathbb{H}^{\mathrm{ND}}_{\mathcal{A}} \in \Sigma^{\mathrm{ND}}_1 \quad \nearrow \quad \nwarrow \quad \Pi^{\mathrm{ND}}_1 \ni \mathrm{INJ}^{\mathrm{ND}}_{\mathcal{A}}$$

$$\nwarrow \quad \Delta^{\mathrm{ND}}_1 \quad \nearrow$$

Thank you very much for your attention!

References

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