

Hierarchies of Decision Problems over Algebraic Structures Defined by Quantifiers

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A shortened version of the slides that were presented at the
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Hierarchies over Algebraic Structures

Introduction

Subject:

- BSS RAM model over first order structures
 - a framework for study of
 - the abstract computability by machines over several structures
 - the uniform abstract decidability and the reducibility of decision problems over algebraic structures
 - on a high abstraction level
 - includes several types of register machines, the Turing machine, and the uniform BSS model of computation over the reals
- hierarchies of undecidable decision problems within this model

Meaning:

- better understanding
 - the structural complexity of decision problems
 - the methods used in the recursion theory
 - the limits of computations over several structures

- The BSS RAM's
 - uniform machines over first order structures
- Halting problems
 - uniformity and codes for machines
- Known hierarchies
 - derived from the arithmetical hierarchy
 - Kleene–Mostowski, Cucker, ...
- A hierarchy over first order structures
 - defined by quantifiers
 - characterized by halting problems
 - complete problems

Computation over Algebraic Structures

The Allowed Instructions (for BSS RAM's)

Computation over $\mathcal{A} = (\underbrace{U_{\mathcal{A}}}_{\text{universe}}; \underbrace{C_{\mathcal{A}}}_{\text{constants}}; \underbrace{f_1, \dots, f_{n_1}}_{\text{operations}}; \underbrace{R_1, \dots, R_{n_2}}_{\text{relations}}, =)$.

Z_1	Z_2	Z_3	Z_4	...
I_1	I_2	I_3	I_4	...

Registers for elements in $U_{\mathcal{A}}$

Registers for indices / addresses

- **Computation** instructions:

$$\ell: Z_j := f_k(Z_{j_1}, \dots, Z_{j_{m_k}})$$

$$\ell: Z_j := d_k$$

$$\text{(e.g. } \ell: Z_j := Z_{j_1} + Z_{j_2})$$

$$(d_k \in U_{\mathcal{A}})$$

- **Branching** instructions:

$\ell: \text{if } Z_i = Z_j \text{ then goto } \ell_1 \text{ else goto } \ell_2$

$\ell: \text{if } R_k(Z_{j_1}, \dots, Z_{j_{n_k}}) \text{ then goto } \ell_1 \text{ else goto } \ell_2$

- **Copy** instructions:

$$\ell: Z_{I_j} := Z_{I_k}$$

- **Index** instructions:

$$\ell: I_j := 1$$

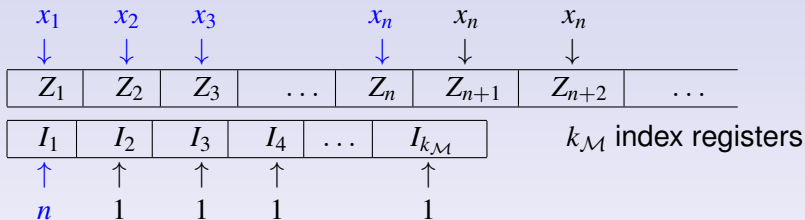
$$\ell: I_j := I_j + 1$$

$\ell: \text{if } I_j = I_k \text{ then goto } \ell_1 \text{ else goto } \ell_2$

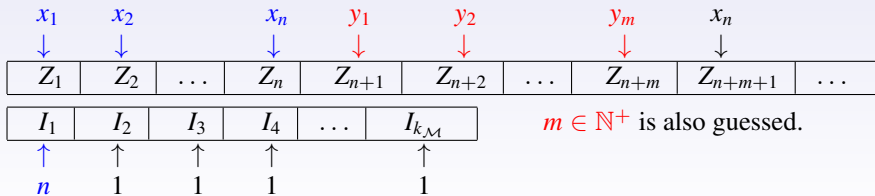
Uniform Computation over Algebraic Structures

Inputs and Outputs for BSS RAM's in $M_{\mathcal{A}}^{[ND]}$

- $U_{\mathcal{A}}^{\infty} =_{\text{df}} \bigcup_{i \geq 1} U_{\mathcal{A}}^i$ — input and output space (for the universe $U_{\mathcal{A}}$)
- Input** of $\vec{x} = (x_1, \dots, x_n) \in U_{\mathcal{A}}^{\infty}$:



- Input** and **guessing** procedures of **nondeterministic** machines:



- Output** of Z_1, \dots, Z_{I_1} .

Two Hierarchies

Analogous to the Arithmetical Hierarchy

- \mathcal{A} is fixed.
- The first hierarchy defined **semantically** by deterministic machines:

$$\begin{aligned}\Sigma_0^0 &= \text{DEC}_{\mathcal{A}} \\ \Pi_n^0 &= \{U_{\mathcal{A}}^\infty \setminus P \mid P \in \Sigma_n^0\} \\ \Delta_n^0 &= \Sigma_n^0 \cap \Pi_n^0 \\ \Sigma_{n+1}^0 &= \{P \subseteq U_{\mathcal{A}}^\infty \mid (\exists Q \in \Pi_n^0)(P \in \text{SDEC}_{\mathcal{A}}^Q)\}\end{aligned}$$

- The second hierarchy defined **syntactically** by formulas:

$$\begin{aligned}\Sigma_0^{\text{ND}} &= \text{DEC}_{\mathcal{A}} \\ \Pi_n^{\text{ND}} &= \{U_{\mathcal{A}}^\infty \setminus P \mid P \in \Sigma_n^{\text{ND}}\} \\ \Delta_n^{\text{ND}} &= \Sigma_n^{\text{ND}} \cap \Pi_n^{\text{ND}} \\ \Sigma_{n+1}^{\text{ND}} &= \{P \subseteq U_{\mathcal{A}}^\infty \mid (\exists Q \in \Pi_n^{\text{ND}}) \\ &\quad \forall \vec{x}(\vec{x} \in P \Leftrightarrow (\exists \vec{y} \in U_{\mathcal{A}}^\infty)((\vec{y} \cdot \vec{x}) \in Q))\}\end{aligned}$$

The Arithmetical Hierarchy (Kleene-Mostowski)

For Turing Machines

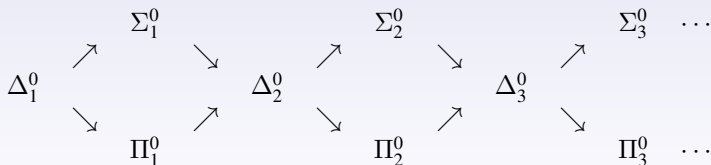
For $(\{0, 1\}; 0, 1; ; =)$, both definitions provide the same hierarchy:

$$\Sigma_{n+1}^0 = \{P \subseteq \{0, 1\}^\infty \mid (\exists Q \in \Pi_n^0)(P \in \text{SDEC}^Q)\}$$

||

$$\Sigma_{n+1}^{\text{ND}} = \{P \subseteq \{0, 1\}^\infty \mid (\exists Q \in \Pi_n^0)$$

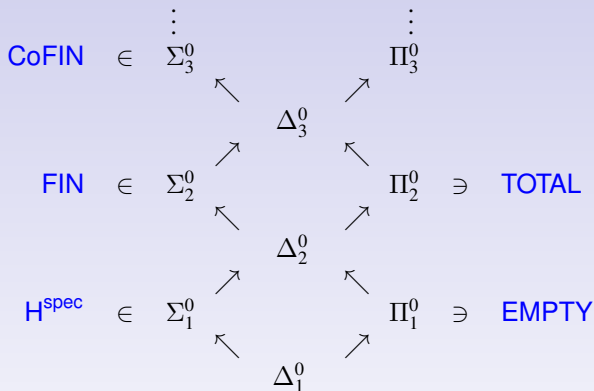
$$\forall \vec{x}(\vec{x} \in P \Leftrightarrow (\exists \vec{y} \in \{0, 1\}^\infty)((\vec{y} \cdot \vec{x}) \in Q))\}$$



“ \rightarrow ” means “strong \subset ”.

Complete Problems in the Arithmetical Hierarchy

For Turing Machines



CoFIN = $\{\text{code}(\mathcal{M}) \mid (\exists n \in \mathbb{N})(\forall \vec{x} \in \{0, 1\}^{(\geq n)})(\mathcal{M}(\vec{x}) \downarrow)\}$

FIN = $\{\text{code}(\mathcal{M}) \mid (\exists n \in \mathbb{N})(\forall \vec{x} \in \{0, 1\}^{(\geq n)})(\mathcal{M}(\vec{x}) \uparrow)\}$

TOTAL = $\{\text{code}(\mathcal{M}) \mid (\forall \vec{x} \in \{0, 1\}^\infty)(\mathcal{M}(\vec{x}) \downarrow)\}$

H^{spec} = $\{\text{code}(\mathcal{M}) \mid \mathcal{M}(\text{code}(\mathcal{M})) \downarrow\}$

EMPTY = $\{\text{code}(\mathcal{M}) \mid (\forall \vec{x} \in \{0, 1\}^\infty)(\mathcal{M}(\vec{x}) \uparrow)\}$ (cp. Soare, Kozen)

Complete Problems in the BSS model (Cucker)

For Computation over the Ring of Reals $\mathbb{R} = (\mathbb{R}; \mathbb{R}; \cdot, +, -, \leq)$

$$\begin{array}{rcccl}
 & & \vdots & & \\
 & & \Sigma_3^{\text{ND}} & \begin{array}{c} \nearrow \\ \nwarrow \end{array} & \Delta_3^{\text{ND}} & \begin{array}{c} \nwarrow \\ \nearrow \end{array} & \Pi_3^{\text{ND}} & & \\
 \text{Suslin's proj. hier.} & = & \Sigma_2^{\text{ND}} & \begin{array}{c} \nearrow \\ \nwarrow \end{array} & \Delta_2^{\text{ND}} & \begin{array}{c} \nwarrow \\ \nearrow \end{array} & \Pi_2^{\text{ND}} & \ni & \text{TOTAL}_{\mathbb{R}}, \text{TOTAL}_{\mathbb{R}}^{\text{ND}} \\
 \text{set of Borel sets} & = & & & \uparrow & & & & \\
 (\subseteq \mathbb{R}^\infty) & & & & & & & &
 \end{array}$$

$$\begin{array}{rcccl}
 & & \vdots & & \vdots & & \\
 & & \Sigma_2^0 & \begin{array}{c} \nearrow \\ \nwarrow \end{array} & \Delta_2^0 & \begin{array}{c} \nwarrow \\ \nearrow \end{array} & \Pi_2^0 & & \\
 \text{FIN}_{\mathbb{R}} & \in & & & & & & & \\
 & & \vdots & & \vdots & & & & \\
 \Sigma_1^{\text{ND}} & = & \Sigma_1^0 & \begin{array}{c} \nearrow \\ \nwarrow \end{array} & \Delta_1^0 & \begin{array}{c} \nwarrow \\ \nearrow \end{array} & \Pi_1^0 & = & \Pi_1^{\text{ND}} \ni \text{INJ}_{\mathbb{R}}
 \end{array}$$

$$\begin{array}{l}
 \text{FIN}_{\mathbb{R}} = \{\text{code}(\mathcal{M}) \mid (\exists n \in \mathbb{N})(\forall \vec{x} \in \mathbb{R}^{(\geq n)})(\mathcal{M}(\vec{x}) \uparrow)\} \\
 \text{INJ}_{\mathbb{R}} = \{\text{code}(\mathcal{M}) \mid (\forall \vec{x}_1, \vec{x}_2 \in \mathbb{R}^\infty)(\mathcal{M}(\vec{x}_1) \downarrow = \mathcal{M}(\vec{x}_2) \downarrow \Rightarrow \vec{x}_1 = \vec{x}_2)\} \\
 \text{TOTAL}_{\mathbb{R}}^{\text{ND}} = \{\text{code}(\mathcal{M}) \mid \mathcal{M} \in \mathbf{M}_{\mathbb{R}}^{\text{ND}} \ \& \ (\forall \vec{x} \in \mathbb{R}^\infty)(\mathcal{M}(\vec{x}) \downarrow)\}
 \end{array}$$

(cp. Cucker)

A Characterization of the First Hierarchy

For BSS RAM's — Computation over Several Structures

For \mathcal{A} :

- a finite number of operations & relations, all elements are constants,
- contains an infinite set effectively enumerable over \mathcal{A} : $\mathbb{N} \subseteq U_{\mathcal{A}}$.

Recall (the definition):

$$\Sigma_0^0 = \text{DEC}_{\mathcal{A}}$$

$$\Pi_n^0 = \{U_{\mathcal{A}}^\infty \setminus P \mid P \in \Sigma_n^0\}$$

$$\Delta_n^0 = \Sigma_n^0 \cap \Pi_n^0$$

$$\Sigma_{n+1}^0 = \{P \subseteq U_{\mathcal{A}}^\infty \mid (\exists Q \in \Pi_n^0)(P \in \text{SDEC}_{\mathcal{A}}^Q)\}$$

$$\Rightarrow \Sigma_{n+1}^0 = \text{SDEC}_{\mathcal{A}}^{\mathbb{H}_{\mathcal{A}}^{(n)}} = \{P \subseteq U_{\mathcal{A}}^\infty \mid P \preceq_1 \mathbb{H}_{\mathcal{A}}^{(n+1)}\}$$

Proposition (G. 2014)

$$\Sigma_{n+1}^0 = \{P \subseteq U_{\mathcal{A}}^\infty \mid (\exists Q \in \Pi_n^0) \forall \vec{x} (\vec{x} \in P \Leftrightarrow (\exists k \in \mathbb{N}) ((\vec{x}.k) \in Q))\}$$

Note: $\mathbb{H}_{\mathcal{A}}^{(0)} = \emptyset$

$\mathbb{H}_{\mathcal{A}}^{(n+1)}$ = Halting problem for BSS RAM's using $\mathbb{H}_{\mathcal{A}}^{(n)}$ as oracle

A Characterization of the Second Hierarchy

For BSS RAM's — Computation over Several Structures

\mathcal{A} : a finite number of operations & relations, all elements $\hat{=}$ constants.

Recall (the definition):

$$\Sigma_0^{\text{ND}} = \text{DEC}_{\mathcal{A}}$$

$$\Pi_n^{\text{ND}} = \{U_{\mathcal{A}}^{\infty} \setminus P \mid P \in \Sigma_n^{\text{ND}}\}$$

$$\Delta_n^{\text{ND}} = \Sigma_n^{\text{ND}} \cap \Pi_n^{\text{ND}}$$

$$\Sigma_{n+1}^{\text{ND}} = \{P \subseteq U_{\mathcal{A}}^{\infty} \mid (\exists Q \in \Pi_n^{\text{ND}}) (\forall \vec{x} (\vec{x} \in P \Leftrightarrow (\exists \vec{y} \in U_{\mathcal{A}}^{\infty}) ((\vec{y} \cdot \vec{x}) \in Q)))\}$$

Proposition (G. 2015)

$$\Sigma_{n+1}^{\text{ND}} = \{P \subseteq U_{\mathcal{A}}^{\infty} \mid (\exists Q \in \Pi_n^{\text{ND}}) (P \in (\text{SDEC}_{\mathcal{A}}^{\text{ND}})^Q)\}$$

$$\Rightarrow \Sigma_{n+1}^{\text{ND}} = (\text{SDEC}_{\mathcal{A}}^{\text{ND}})^{(\mathbb{H}_{\mathcal{A}}^{\text{ND}})^{(n)}} = \{P \subseteq U_{\mathcal{A}}^{\infty} \mid P \preceq_1 (\mathbb{H}_{\mathcal{A}}^{\text{ND}})^{(n+1)}\}$$

Note: $(\mathbb{H}_{\mathcal{A}}^{\text{ND}})^{(0)} = \emptyset$

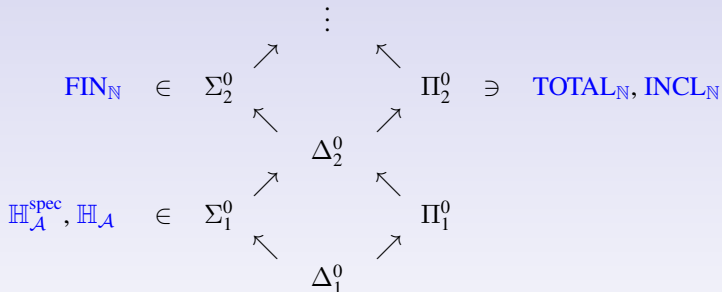
$(\mathbb{H}_{\mathcal{A}}^{\text{ND}})^{(n+1)} = \text{Halting p. for ND-machines using } (\mathbb{H}_{\mathcal{A}}^{\text{ND}})^{(n)} \text{ as oracle}$

Complete Problems in the First Hierarchy

For BSS RAM's — Computation over Several Structures

For \mathcal{A} :

- a finite number of operations & relations, all elements are constants,
- contains an infinite set effectively enumerable over \mathcal{A} : $\mathbb{N} \subseteq U_{\mathcal{A}}$.



$FIN_{\mathbb{N}} = \{\text{code}(\mathcal{M}) \in U_{\mathcal{A}}^{\infty} \mid |H_{\mathcal{M}} \cap \mathbb{N}^{\infty}| < \infty\}$ ($H_{\mathcal{M}}$ = halting set)

$TOTAL_{\mathbb{N}} = \{\text{code}(\mathcal{M}) \in U_{\mathcal{A}}^{\infty} \mid (\forall \vec{x} \in \mathbb{N}^{\infty})(\mathcal{M}(\vec{x}) \downarrow)\}$

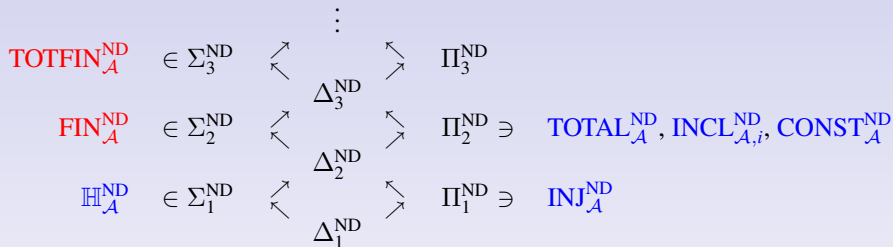
$INCL_{\mathbb{N}} = \{(\text{code}(\mathcal{M}) . \text{code}(\mathcal{N})) \in U_{\mathcal{A}}^{\infty} \mid (H_{\mathcal{M}} \cap \mathbb{N}^{\infty}) \subseteq (H_{\mathcal{N}} \cap \mathbb{N}^{\infty})\}$

$IH_{\mathcal{A}}^{[spec]} \hat{=} \text{Halting problems for BSS RAM's over } \mathcal{A}$ (cp. Gaßner)

Complete Problems in the Second Hierarchy (Blue)

For BSS RAM's — Computation over Several Structures

\mathcal{A} : a finite number of operations & relations, all elements $\hat{=}$ constants.



$$\text{TOTAL}_{\mathcal{A}}^{\text{ND}} = \{\text{code}(\mathcal{M}) \mid (\forall \vec{x} \in \mathbb{R}^\infty)(\mathcal{M}(\vec{x}) \downarrow)\} \quad (\mathcal{M} \in \mathbf{M}_{\mathcal{A}}^{\text{ND}})$$

$$\text{INJ}_{\mathcal{A}}^{\text{ND}} = \{\text{code}(\mathcal{M}) \mid \mathcal{M} \text{ computes a/an [super] injective function}\}$$

$$\text{CONST}_{\mathcal{A}}^{\text{ND}} = \{\text{code}(\mathcal{M}) \mid \mathcal{M} \text{ computes a total constant function}\}$$

$$\text{FIN}_{\mathcal{A}}^{\text{ND}} = \{\text{code}(\mathcal{M}) \mid (\forall i \in \mathbb{N} \setminus I)(H_{\mathcal{M}} \cap U_{\mathcal{A}}^i = \emptyset) \text{ for some } |I| < \omega\}$$

$$\text{TOTFIN}_{\mathcal{A}}^{\text{ND}} = \{\text{code}(\mathcal{M}) \mid (\forall i \in \mathbb{N} \setminus I)(H_{\mathcal{M}} \cap U_{\mathcal{A}}^i \neq U_{\mathcal{A}}^i) \text{ for some } |I| < \omega\}$$

$$\text{IH}_{\mathcal{A}}^{\text{ND}} = \text{Halting problem for ND-machines over } \mathcal{A} \text{ (in } \mathbf{M}_{\mathcal{A}}^{\text{ND}})$$

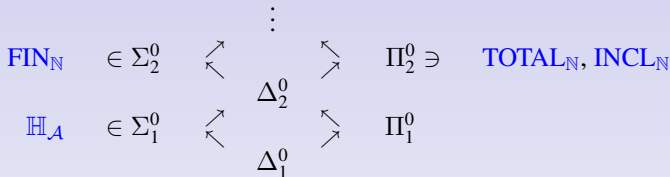
$$\text{INCL}_{\mathcal{A},i}^{\text{ND}} = \{(\text{code}(\mathcal{M}) . \text{code}(\mathcal{N})) \mid (\mathcal{M}, \mathcal{N}) \in \mathbf{M}_{\mathcal{A},i} \times \mathbf{M}_{\mathcal{A}}^{\text{ND}} \ \& \ H_{\mathcal{M}} \subseteq H_{\mathcal{N}}\}$$

$$\mathbf{M}_{\mathcal{A},1} = \mathbf{M}_{\mathcal{A}}, \quad \mathbf{M}_{\mathcal{A},2} = \mathbf{M}_{\mathcal{A}}(\text{IH}_{\mathcal{A}}), \quad \mathbf{M}_{\mathcal{A},3} = \mathbf{M}_{\mathcal{A}}(\text{IH}_{\mathcal{A}}^{\text{ND}}), \quad \mathbf{M}_{\mathcal{A},4} = \mathbf{M}_{\mathcal{A}}^{\text{ND}}(\text{IH}_{\mathcal{A}}^{\text{ND}})$$

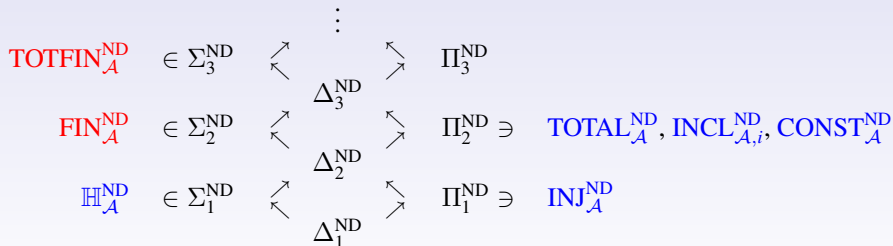
Summary

For BSS RAM's — Computation over Several Structures

1st hierarchy:



2nd hierarchy:



Thank you very much for your attention!

References

L. BLUM, M. SHUB, and S. SMALE: “On a theory of computation and complexity over the real numbers: NP-completeness, recursive functions and universal machines” (1989)

F. CUCKER: “The arithmetical hierarchy over the reals” (1992)

C. GASSNER: “Computation over algebraic structures and a classification of undecidable problems” (to appear in: MSCS)

D. C. KOZEN: “Theory of computation” (2006)

R. I. SOARE: “Recursively enumerable sets and degrees: a study of computable functions and computably generated sets” (1987)

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