Moschovakis Operators for BSS RAM’s

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Definitions and Motivation

Operators for BSS RAM's

(Kleene's Operator, Moschovakis' Operator, Det. and Nondet. Moschovakis Operators)

$\Downarrow$ Stephen C. Kleene

Recursion Theory based on recursion and μ-operator

("Kleene's Operator" for the Peano structure, deterministic)

$\Downarrow$ Yiannis N. Moschovakis

Generalized Recursion Theory based on recursion and ν-operator

("Moschovakis' Operator" for several structures, nondeterministic)

$\Downarrow$ Gaßner

ν-operators for BSS RAM's over arbitrary mathematical structures

(nondeterministic "Moschovakis Operators")

$\Downarrow$

Deterministic "Moschovakis Operators" for some structures:

ν_{min}-operators for BSS RAM's (similar to Kleene's μ-operator),

ν_{inf}-operators for BSS RAM's, ν_{lim}-operators for BSS RAM's, ...

$\Downarrow$

One goal is to investigate the power of ν_{min}, ν_{inf}, ...

Computation by BSS RAM's over Algebraic Structures

(The Machines and the Allowed Instructions)

Computation over $\mathcal{A} = (\underbrace{U_\mathcal{A}}_{\text{universe}} ; \underbrace{C_\mathcal{A}}_{\text{constants}} ; \underbrace{f_1, \ldots, f_{n_1}}_{\text{operations}} ; \underbrace{R_1, \ldots, R_{n_2}, =}_{\text{relations}})$.

Z_1	Z_2	Z_3	Z_4	Z_5	$\ldots$

Registers for elements in $U_\mathcal{A}$

I_1	I_2	I_3	I_4	$\ldots$	$I_{k_\mathcal{M}}$

Registers for indices in $\mathbb{N}$

- Computation instructions:

  $\ell\colon\ Z_j := f_k(Z_{j_1}, \ldots, Z_{j_{m_k}})$ (e.g. $\ell\colon\ Z_j := Z_{j_1} + Z_{j_2}$)

  $\ell\colon\ Z_j := d_k$ $(d_k \in C_\mathcal{A} \subseteq U_\mathcal{A})$

- Branching instructions:

  $\ell\colon$ if $Z_i = Z_j$ then goto ℓ_1 else goto ℓ_2

  $\ell\colon$ if $R_k(Z_{j_1}, \ldots, Z_{j_{n_k}})$ then goto ℓ_1 else goto ℓ_2

- Copy instructions:

  $\ell\colon\ Z_{I_j} := Z_{I_k}$

- Index instructions:

  $\ell\colon\ I_j := 1$

  $\ell\colon\ I_j := I_j + 1$

  $\ell\colon$ if $I_j = I_k$ then goto ℓ_1 else goto ℓ_2

Uniform Computation over Algebraic Structures
(Input and Output Procedures)

- $U_{\mathcal{A}}$ is the universe of $\mathcal{A}$
- Input and output space: $U_{\mathcal{A}}^{\infty} =_{\mathrm{df}} \bigcup_{i \geq 1} U_{\mathcal{A}}^{i}$

- Input of $\vec{x} = (x_1, \ldots, x_n) \in U_{\mathcal{A}}^{\infty}$:

x_1	x_2	x_3	x_4		x_n	x_n	x_n	
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$		$\downarrow$	$\downarrow$	$\downarrow$	
Z_1	Z_2	Z_3	Z_4	$\ldots$	Z_n	Z_{n+1}	Z_{n+2}	$\ldots$

n	1	1	1		1
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$		$\downarrow$
I_1	I_2	I_3	I_4	$\ldots$	$I_{k_{\mathcal{M}}}$

- Output of $Z_1, \ldots, Z_{I_1}$.

[ν-]Semi-Decidability
(The Definitions)

$P \subseteq U_{\mathcal{A}}^{\infty}$ is a *decision problem*.

$P \subseteq U_{\mathcal{A}}^{\infty}$ is *semi-decidable* if there is a BSS RAM $\mathcal{M}$ such that

$$\vec{x} \in P \Leftrightarrow \mathcal{M} \text{ halts on } \vec{x}.$$

$P \subseteq U_{\mathcal{A}}^{\infty}$ is *ν-semi-decidable* if there is a ν-oracle BSS RAM $\mathcal{M}$ such that

$$\vec{x} \in P \Leftrightarrow \mathcal{M} \text{ halts on } \vec{x}.$$

...

ν-oracle BSS RAM $\mathcal{M}$ = BSS RAM $\mathcal{M}$ using operator ν

...

μ-Oracle BSS RAM's with μ-Operators for $\mathbb{N} \subseteq U_{\mathcal{A}}$
(Kleene's Operator)

- $\mathcal{A}$ fixed, $U_{\mathcal{A}}$ contains an effectively enumerable set denoted by $\mathbb{N}$,
  $a = 1$, $b = 0$.
  $f : U_{\mathcal{A}}^{\infty} \to \{a, b\}$ computable over $\mathcal{A}$.

- Kleene's operator for $\mathcal{A}$:

  $\mu[f](x_1, \ldots, x_n)$
  $=_{\text{df}} \min\{k \in \mathbb{N} \mid f(x_1, \ldots, x_n, k) = 1 \;\&\; f(x_1, \ldots, x_n, l) \downarrow \text{ for } l < k\}$

$$\begin{array}{c} \quad z_1 \;\cdots\; z_n \\ \quad\downarrow \qquad \downarrow \\ \ell :\; Z_j := \mu[f](Z_1, \ldots, Z_{I_1}), \qquad \text{if } I_1 = n \end{array}$$

no minimum $\Rightarrow$ the machine loops forever

Properties

$\mathbb{N} = U_{\mathcal{A}}$. $\Rightarrow$ *Any μ-semi-decidable problem is semi-decidable over $\mathcal{A}$.*

ν-Oracle BSS RAM's for Structures with a and b
(Moschovakis' Operator)

- $\mathcal{A}$ is fixed. a, b are constants of $\mathcal{A}$.
  $f : U_{\mathcal{A}}^{\infty} \to \{a, b\}$ computable over $\mathcal{A}$.

- Moschovakis' operator for $\mathcal{A}$:

$$\nu[f](x_1, \ldots, x_n)$$
$$=_{\text{df}} \{y_1 \in U_{\mathcal{A}} \mid (\exists (y_2, \ldots, y_m) \in U_{\mathcal{A}}^{\infty})(f(x_1, \ldots, x_n, \underbrace{y_1, \ldots, y_m}_{\vec{y} \in U_{\mathcal{A}}^{\infty}}) = a)\}$$

NONDETERMINISTIC!

$$\begin{array}{c} \quad\quad\quad\quad\quad\quad z_1 \quad \cdots \quad z_n \\ \quad\quad\quad\quad\quad\quad \downarrow \quad\quad\quad \downarrow \\ \ell : \; Z_j := \nu[f](Z_1, \ldots, Z_{I_1}) \end{array}$$

$\nu[f](z_1, \ldots, z_n) \neq \emptyset \Rightarrow Z_j$ contains some $z \in \nu[f](z_1, \ldots, z_n)$.
$\nu[f](z_1, \ldots, z_n) = \emptyset \Rightarrow$ no stop (the machine loops forever).

Nondeterministic ν-Oracle BSS RAM's
(Guessing Solutions with Moschovakis' Operator)

- $(z_1, \ldots, z_{n+m}) \overset{f}{\mapsto} w \in \{a, b\}$ computable over $\mathcal{A}$.
- Nondeterministic computation with Moschovakis' operator:

$$\underset{\substack{\downarrow \\ y_1}}{Z_{I_1+1}} := \nu[f](\overset{\substack{x_1 \\ \downarrow}}{Z_1}, \ldots, \overset{\substack{x_n \\ \downarrow}}{Z_{I_1}}); \quad \underset{\substack{\downarrow \\ y_2}}{Z_{I_1+2}} := \nu[f](\overset{\substack{x_1 \\ \downarrow}}{Z_1}, \ldots, \overset{\substack{x_n \\ \downarrow}}{Z_{I_1}}, \overset{\substack{y_1 \\ \downarrow}}{Z_{I_1+1}}) \quad \ldots$$

$$\Rightarrow f(x_1, \ldots, x_n, y_1, \ldots, y_m) = a$$

ν-Oracle BSS RAM's versus ν_m-Oracle BSS RAM's
(Motivation for Deterministic Uniform Operators: Computable Choice Functions?)

- $\mathcal{A} = (\mathbb{N}; \mathbb{N}; ; =)$.
- $f(x_1, \ldots, x_n) := \begin{cases} 1 & \text{if } x_i \neq x_j \text{ for all } i, j \text{ with } i \neq j, \\ 0 & \text{otherwise.} \end{cases}$

$$\underbrace{\ell : \; Z_j := \nu[f](Z_1, \ldots, Z_{I_1})}_{(1)}, \quad \text{with } z_1 \cdots z_n \downarrow \text{ into } Z_1, \ldots, Z_{I_1}$$

$$\underbrace{\ell : \; Z_j := \nu_m[f](Z_1, \ldots, Z_m)}_{(2)}, \quad \text{with } z_1 \cdots z_m \downarrow \text{ into } Z_1, \ldots, Z_m$$

(1) We get a $z \in \mathbb{N} \setminus \{z_1, \ldots, z_n\}$ for any n and $(z_1, \ldots, z_n)$ with $z_i \neq z_{i+k}$.
(2) We get a $z \in \mathbb{N} \setminus \{z_1, \ldots, z_m\}$ for $(z_1, \ldots, z_m) \in U^m_{\mathcal{A}}$ with $z_i \neq z_{i+k}$.

Properties

(1) For the correspondence $(z_1, \ldots, z_n) \mapsto \mathbb{N} \setminus \{z_1, \ldots, z_n\}$ we do not have a computable choice function.
(2) For the correspondence $(z_1, \ldots, z_m) \mapsto \mathbb{N} \setminus \{z_1, \ldots, z_m\}$ we have a choice function which can be computed by means of $m + 1$ constants.

$\nu_{\min}$-Oracle BSS RAM's versus Simple BSS RAM's
(Motivation for Deterministic Uniform Operator $\nu_{\min}$)

- $\mathcal{A} = (\mathbb{N}\times\mathbb{N}; \mathbb{N}\times\{0\}; s; \leq_{\text{lexi}})$ with $s(n,m) = (n, m+1)$. $\Rightarrow$ $U_{\mathcal{A}}$ is not enumerable over $\mathcal{A}$. $\leq_{\text{lexi}}$ is a decidable well-ordering. $a = (1,0), b = (0,0)$ are constants of $\mathcal{A}$.
- Moschovakis operator $\nu_{\min}$:
  $$\nu_{\min}[f](x_1,\ldots,x_n) =_{\text{df}} \min\{y_1 \in U_{\mathcal{A}} \mid (\exists (y_2,\ldots,y_m) \in U_{\mathcal{A}}^{\infty})(f(x_1,\ldots,x_n,\underbrace{y_1,\ldots,y_m}_{\vec{y}\in U_{\mathcal{A}}^{\infty}}) = a)\}$$
- $f((n_1,m_1),(n_2,m_2)) := \begin{cases} a & \text{if } n_1 = n_2, \\ \uparrow & \text{otherwise.} \end{cases}$ $\qquad g((n,m)) := (n,0)$

Properties

f is computable by a BSS RAM over $\mathcal{A}$.
g is not computable by a BSS RAM over $\mathcal{A}$,
but it is computable by a $\nu_{\min}$-oracle BSS RAM over $\mathcal{A}$.
$\mathbb{N}\times\{0\}$ is not semi-decidable by a BSS RAM over $\mathcal{A}$,
but it is $\nu_{\min}$-semi-decidable by a $\nu_{\min}$-oracle BSS RAM over $\mathcal{A}$.