

Moschovakis Operators for BSS RAM's

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Definitions and Motivation

Operators for BSS RAM's

(Kleene's Operator, Moschovakis' Operator, Det. and Nondet. Moschovakis Operators)

↓ Stephen C. Kleene

Recursion Theory based on recursion and μ -operator

(“Kleene's Operator” for the Peano structure, deterministic)

↓ Yiannis N. Moschovakis

Generalized Recursion Theory based on recursion and ν -operator

(“Moschovakis' Operator” for several structures, nondeterministic)

↓ Gaßner

ν -operators for BSS RAM's over arbitrary mathematical structures

(nondeterministic “Moschovakis Operators”)

↓

Deterministic “Moschovakis Operators” for some structures:

ν_{\min} -operators for BSS RAM's (similar to Kleene's μ -operator),

ν_{\inf} -operators for BSS RAM's, ν_{\lim} -operators for BSS RAM's, ...

↓

One goal is to investigate the power of ν_{\min} , ν_{\inf} , ...

Computation by BSS RAM's over Algebraic Structures

(The Machines and the Allowed Instructions)

Computation over $\mathcal{A} = (\underbrace{U_{\mathcal{A}}}_{\text{universe}}; \underbrace{C_{\mathcal{A}}}_{\text{constants}}; \underbrace{f_1, \dots, f_{n_1}}_{\text{operations}}; \underbrace{R_1, \dots, R_{n_2}}_{\text{relations}} =)$.

Z_1	Z_2	Z_3	Z_4	Z_5	...
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Registers for elements in $U_{\mathcal{A}}$

I_1	I_2	I_3	I_4	...	$I_{k_{\mathcal{M}}}$
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Registers for indices in \mathbb{N}

- **Computation** instructions:

$$\ell: Z_j := f_k(Z_{j_1}, \dots, Z_{j_{m_k}})$$

$$\text{(e.g. } \ell: Z_j := Z_{j_1} + Z_{j_2})$$

$$\ell: Z_j := d_k$$

$$(d_k \in C_{\mathcal{A}} \subseteq U_{\mathcal{A}})$$

- **Branching** instructions:

$$\ell: \text{if } Z_i = Z_j \text{ then goto } \ell_1 \text{ else goto } \ell_2$$

$$\ell: \text{if } R_k(Z_{j_1}, \dots, Z_{j_{n_k}}) \text{ then goto } \ell_1 \text{ else goto } \ell_2$$

- **Copy** instructions:

$$\ell: Z_{I_j} := Z_{I_k}$$

- **Index** instructions:

$$\ell: I_j := 1$$

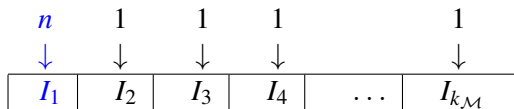
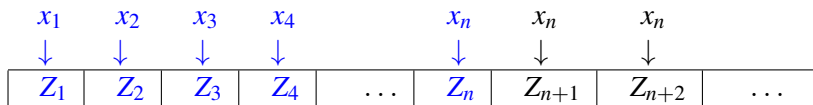
$$\ell: I_j := I_j + 1$$

$$\ell: \text{if } I_j = I_k \text{ then goto } \ell_1 \text{ else goto } \ell_2$$

Uniform Computation over Algebraic Structures

(Input and Output Procedures)

- $U_{\mathcal{A}}$ is the universe of \mathcal{A}
- Input and output space: $U_{\mathcal{A}}^{\infty} =_{\text{df}} \bigcup_{i \geq 1} U_{\mathcal{A}}^i$
- **Input** of $\vec{x} = (x_1, \dots, x_n) \in U_{\mathcal{A}}^{\infty}$:



- **Output** of Z_1, \dots, Z_{I_1} .

$[\nu-]$ Semi-Decidability

(The Definitions)

$P \subseteq U_{\mathcal{A}}^{\infty}$ is a *decision problem*.

$P \subseteq U_{\mathcal{A}}^{\infty}$ is *semi-decidable* if there is a BSS RAM \mathcal{M} such that

$$\vec{x} \in P \Leftrightarrow \mathcal{M} \text{ halts on } \vec{x}.$$

$P \subseteq U_{\mathcal{A}}^{\infty}$ is *ν -semi-decidable* if there is a ν -oracle BSS RAM \mathcal{M} such that

$$\vec{x} \in P \Leftrightarrow \mathcal{M} \text{ halts on } \vec{x}.$$

...

ν -oracle BSS RAM \mathcal{M} = BSS RAM \mathcal{M} using operator ν

...

μ -Oracle BSS RAM's with μ -Operators for $\mathbb{N} \subseteq U_{\mathcal{A}}$ (Kleene's Operator)

- \mathcal{A} fixed, $U_{\mathcal{A}}$ contains an **effectively enumerable set** denoted by \mathbb{N} ,
 $a = 1, b = 0$.

$f : U_{\mathcal{A}}^{\infty} \rightarrow \{a, b\}$ computable over \mathcal{A} .

- **Kleene's operator** for \mathcal{A} :

$\mu[f](x_1, \dots, x_n)$

$=_{\text{df}} \min\{k \in \mathbb{N} \mid f(x_1, \dots, x_n, k) = 1 \ \& \ f(x_1, \dots, x_n, l) \downarrow \text{ for } l < k\}$

$$\begin{array}{ccc} z_1 & \cdots & z_n \\ \downarrow & & \downarrow \\ \ell : Z_j := \mu[f](Z_1, \dots, Z_{I_1}), & & \text{if } I_1 = n \end{array}$$

no minimum \Rightarrow the machine loops forever

Properties

$\mathbb{N} = U_{\mathcal{A}} \Rightarrow$ Any μ -semi-decidable problem is semi-decidable over \mathcal{A} .

ν -Oracle BSS RAM's for Structures with a and b

(Moschovakis' Operator)

- \mathcal{A} is fixed. a, b are constants of \mathcal{A} .
 $f : U_{\mathcal{A}}^{\infty} \rightarrow \{a, b\}$ computable over \mathcal{A} .

- **Moschovakis' operator** for \mathcal{A} :

$$\nu[f](x_1, \dots, x_n) \\ =_{\text{df}} \{y_1 \in U_{\mathcal{A}} \mid (\exists (y_2, \dots, y_m) \in U_{\mathcal{A}}^{\infty}) (f(x_1, \dots, x_n, \underbrace{y_1, \dots, y_m}_{\vec{y} \in U_{\mathcal{A}}^{\infty}}) = a)\}$$

NONDETERMINISTIC!

$$\begin{array}{ccc} z_1 & \cdots & z_n \\ \downarrow & & \downarrow \\ \ell : Z_j & := & \nu[f](Z_1, \dots, Z_{I_j}) \end{array}$$

$\nu[f](z_1, \dots, z_n) \neq \emptyset \Rightarrow Z_j$ contains some $z \in \nu[f](z_1, \dots, z_n)$.
 $\nu[f](z_1, \dots, z_n) = \emptyset \Rightarrow$ no stop (the machine loops forever).

Nondeterministic ν -Oracle BSS RAM's

(Guessing Solutions with Moschovakis' Operator)

- $(z_1, \dots, z_{n+m}) \stackrel{f}{\mapsto} w \in \{a, b\}$ computable over \mathcal{A} .
- Nondeterministic computation with **Moschovakis' operator**:

$$\begin{array}{ccc} x_1 & \cdots & x_n \\ \downarrow & & \downarrow \\ Z_{I_1+1} := \nu[f](Z_1, \dots, Z_{I_1}); & Z_{I_1+2} := \nu[f](Z_1, \dots, Z_{I_1}, Z_{I_1+1}) & \dots \\ \downarrow & \downarrow & \\ y_1 & y_2 & \end{array}$$

$$\Rightarrow f(x_1, \dots, x_n, y_1, \dots, y_m) = a$$

ν -Oracle BSS RAM's versus ν_m -Oracle BSS RAM's

(Motivation for Deterministic Uniform Operators: Computable Choice Functions?)

- $\mathcal{A} = (\mathbb{N}; \mathbb{N}; ; =)$.
- $f(x_1, \dots, x_n) := \begin{cases} 1 & \text{if } x_i \neq x_j \text{ for all } i, j \text{ with } i \neq j, \\ 0 & \text{otherwise.} \end{cases}$

$$\underbrace{\ell : Z_j := \nu[f](Z_1, \dots, Z_n)}_{(1)}$$

$z_1 \quad \dots \quad z_n$
 $\downarrow \quad \quad \quad \downarrow$

$$\underbrace{\ell : Z_j := \nu_m[f](Z_1, \dots, Z_m)}_{(2)}$$

$z_1 \quad \dots \quad z_m$
 $\downarrow \quad \quad \quad \downarrow$

- (1) We get a $z \in \mathbb{N} \setminus \{z_1, \dots, z_n\}$ for any n and (z_1, \dots, z_n) with $z_i \neq z_{i+k}$.
- (2) We get a $z \in \mathbb{N} \setminus \{z_1, \dots, z_m\}$ for $(z_1, \dots, z_m) \in U_{\mathcal{A}}^m$ with $z_i \neq z_{i+k}$.

Properties

- (1) For the correspondence $(z_1, \dots, z_n) \mapsto \mathbb{N} \setminus \{z_1, \dots, z_n\}$ we do not have a computable choice function.
- (2) For the correspondence $(z_1, \dots, z_m) \mapsto \mathbb{N} \setminus \{z_1, \dots, z_m\}$ we have a choice function which can be computed by means of $m + 1$ constants.

ν_{\min} -Oracle BSS RAM's versus Simple BSS RAM's

(Motivation for Deterministic Uniform Operator ν_{\min})

- $\mathcal{A} = (\mathbb{N} \times \mathbb{N}; \mathbb{N} \times \{0\}; s; \leq_{\text{lexi}})$ with $s(n, m) = (n, m + 1)$. $\Rightarrow U_{\mathcal{A}}$ is not enumerable over \mathcal{A} . \leq_{lexi} is a decidable well-ordering. $a = (1, 0), b = (0, 0)$ are constants of \mathcal{A} .

- **Moschovakis operator** ν_{\min} :

$$\nu_{\min}[f](x_1, \dots, x_n) =_{\text{df}} \min\{y_1 \in U_{\mathcal{A}} \mid (\exists (y_2, \dots, y_m) \in U_{\mathcal{A}}^{\infty})(f(x_1, \dots, x_n, \underbrace{y_1, \dots, y_m}_{\vec{y} \in U_{\mathcal{A}}^{\infty}}) = a)\}$$

- $f((n_1, m_1), (n_2, m_2)) := \begin{cases} a & \text{if } n_1 = n_2, \\ \uparrow & \text{otherwise.} \end{cases} \quad g((n, m)) := (n, 0)$

Properties

f is computable by a BSS RAM over \mathcal{A} .

g is not computable by a BSS RAM over \mathcal{A} ,

but it is computable by a ν_{\min} -oracle BSS RAM over \mathcal{A} .

$\mathbb{N} \times \{0\}$ is not semi-decidable by a BSS RAM over \mathcal{A} ,

but it is ν_{\min} -semi-decidable by a ν_{\min} -oracle BSS RAM over \mathcal{A} .