BSS RAM's with Operators for Several Measures

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Kloster (Hiddensee) 2016

$\Downarrow$ Stephen C. Kleene

Recursion Theory based on recursion and μ-operator

("Kleene's Operator" for the Peano structure, deterministic)

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Generalized Recursion Theory based on recursion and ν-operator

("Moschovakis' Operator" for several structures, nondeterministic)

Operators for BSS RAM's

(Kleene's Operator, Moschovakis' Operator, Det. and Nondet. Moschovakis Operators)

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ν-Operators for BSS RAM's over arbitrary mathematical structures

(nondeterministic "Moschovakis Operators")

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Deterministic operators for BSS RAM's over certain structures

ν_{min}-operators (similar to Kleene's μ-operator), ν_{inf}-operators,
ν_{lim}-operators, ν_{Lebes}-operators, ν_{δ_p}-operators, ν_{count}-operators, . . .

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⇒ A goal: investigate the power of

- ν_{min} ν_{inf}, ν_{lim}, . . . (with P. F. V. Vizcaíno)
- ν_{Lebes}, ν_{δ_p}, ν_{count}, . . .

Computation over $\mathcal{A} = (\underbrace{U_\mathcal{A}}_{\text{universe}} ; \underbrace{C_\mathcal{A}}_{\text{constants}} ; \underbrace{f_1, \ldots, f_{n_1}}_{\text{operations}} ; \underbrace{R_1, \ldots, R_{n_2}, =}_{\text{relations}})$.

Z_1	Z_2	Z_3	Z_4	Z_5	$\ldots$

Registers for elements in $U_\mathcal{A}$

I_1	I_2	I_3	I_4	$\ldots$	$I_{k_\mathcal{M}}$

Registers for indices in $\mathbb{N}$

- Computation instructions:

  $\ell\colon\ Z_j := f_k(Z_{j_1}, \ldots, Z_{j_{m_k}})$ (e.g. $\ell\colon\ Z_j := Z_{j_1} + Z_{j_2}$)

  $\ell\colon\ Z_j := d_k$ $(d_k \in C_\mathcal{A} \subseteq U_\mathcal{A})$

- Branching instructions:

  $\ell\colon$ if $Z_i = Z_j$ then goto ℓ_1 else goto ℓ_2

  $\ell\colon$ if $R_k(Z_{j_1}, \ldots, Z_{j_{n_k}})$ then goto ℓ_1 else goto ℓ_2

- Copy instructions:

  $\ell\colon\ Z_{I_j} := Z_{I_k}$

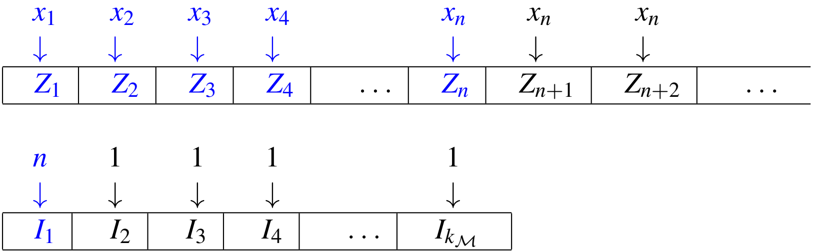
- Index instructions:

  $\ell\colon\ I_j := 1$

  $\ell\colon\ I_j := I_j + 1$

  $\ell\colon$ if $I_j = I_k$ then goto ℓ_1 else goto ℓ_2

- $U_{\mathcal{A}}$ is the universe of $\mathcal{A}$
- Input and output space: $U_{\mathcal{A}}^{\infty} =_{\mathrm{df}} \bigcup_{i \geq 1} U_{\mathcal{A}}^{i}$
- Input of $\vec{x} = (x_1, \ldots, x_n) \in U_{\mathcal{A}}^{\infty}$:



- Output of $Z_1, \ldots, Z_{I_1}$.

$P \subseteq U_{\mathcal{A}}^{\infty}$ is a *decision problem*.

$P \subseteq U_{\mathcal{A}}^{\infty}$ is *semi-decidable* if there is a BSS RAM $\mathcal{M}$ such that

$$\vec{x} \in P \Leftrightarrow \mathcal{M} \text{ halts on } \vec{x}.$$

$P \subseteq U_{\mathcal{A}}^{\infty}$ is *ν-semi-decidable* if there is a ν-oracle BSS RAM $\mathcal{M}$ such that

$$\vec{x} \in P \Leftrightarrow \mathcal{M} \text{ halts on } \vec{x}.$$

. . .

ν-oracle BSS RAM $\mathcal{M}$ = BSS RAM $\mathcal{M}$ using operator ν

. . .

μ-Oracle BSS RAM's with μ-Operators for $\mathbb{N} \subseteq U_{\mathcal{A}}$
(Kleene's Operator)

$\mathcal{A}$ fixed, $U_{\mathcal{A}}$ contains an effectively enumerable set denoted by $\mathbb{N}$,
$a = 1$, $b = 0$.
$f : U_{\mathcal{A}}^{\infty} \to \{a, b\}$ partial function, computable over $\mathcal{A}$ (or $f : U_{\mathcal{A}}^{\infty} \to U_{\mathcal{A}}^{\infty}$).

Definition (Kleene's operator for $\mathcal{A}$)

$\mu[f](x_1, \ldots, x_n)$
$=_{\text{df}} \min\{k \in \mathbb{N} \mid f(x_1, \ldots, x_n, k) = 1 \;\&\; f(x_1, \ldots, x_n, l) \downarrow \text{ for } l < k\}$

Definition (Oracle Instruction with Kleene's operator)

$$\begin{array}{ll} \qquad\qquad\quad z_1 \;\cdots\; z_n & \\ \qquad\qquad\quad \downarrow \qquad\; \downarrow & \\ \ell : \; Z_j := \mu[f](Z_1, \ldots, Z_{I_1}), & \text{if } I_1 = n \end{array}$$

no minimum $\Rightarrow$ the machine loops forever

Properties

$\mathbb{N} = U_{\mathcal{A}}$. $\Rightarrow$ *Any μ-semi-decidable problem is semi-decidable over $\mathcal{A}$.*

ν-Oracle BSS RAM's for Structures with a and b
(Moschovakis' Operator)

$\mathcal{A}$ is fixed. a, b are constants of $\mathcal{A}$.
$f : U_{\mathcal{A}}^{\infty} \to \{a, b\}$ partial function, computable over $\mathcal{A}$.

Definition (Moschovakis' operator for $\mathcal{A}$)

$$\nu[f](x_1, \ldots, x_n)$$
$$=_{\mathrm{df}} \{y_1 \in U_{\mathcal{A}} \mid (\exists(y_2, \ldots, y_m) \in U_{\mathcal{A}}^{\infty})(f(x_1, \ldots, x_n, \underbrace{y_1, y_2, \ldots, y_m}_{\vec{y} \in U_{\mathcal{A}}^{\infty}}) = a)\}$$

Definition (Oracle instruction with Moschovakis' operator)

$$\begin{array}{ll} & \;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\; z_1 \;\cdots\; z_n \\ & \;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\;\; \downarrow \;\;\;\;\;\; \downarrow \\ \text{NONDETERMINISTIC!} & \ell : \; Z_j := \nu[f](Z_1, \ldots, Z_{I_1}) \end{array}$$

$\nu[f](z_1, \ldots, z_n) \neq \emptyset \quad \Rightarrow \quad Z_j$ contains some $z \in \nu[f](z_1, \ldots, z_n)$.
$\nu[f](z_1, \ldots, z_n) = \emptyset \quad \Rightarrow \quad$ no stop (the machine loops forever).

Nondeterministic ν-Oracle BSS RAM's
(Guessing Solutions with Moschovakis' Operator)

$f : U_{\mathcal{A}}^{\infty} \to \{a, b\}$ partial function, computable over $\mathcal{A}$.

$(z_1, \ldots, z_{n+m}) \overset{f}{\mapsto} w \in \{a, b\}$ computable over $\mathcal{A}$.

Properties

Nondeterministic computation with Moschovakis' operator:

$$
\begin{array}{ccccccc}
 & x_1 \;\cdots\; x_n & & & x_1 \;\cdots\; x_n & {\color{red}y_1} & \\
 & \downarrow \qquad \downarrow & & & \downarrow \qquad \downarrow & {\color{red}\downarrow} & \\
Z_j := \nu[f](Z_1, \ldots, Z_{I_1}); \ldots & & Z_j := \nu[f](Z_1, \ldots, Z_{I_1 - 1}, Z_{I_1}) & & & & \ldots \\
\downarrow & & \downarrow & & & & \\
{\color{red}y_1} & & {\color{red}y_2} & & & &
\end{array}
$$

$\Rightarrow f(x_1, \ldots, x_n, {\color{red}y_1, \ldots, y_m}) = a$

ν-Oracle BSS RAM's versus ν_m-Oracle BSS RAM's
(Motivation for Deterministic Uniform Operators: Computable Choice Functions?)

$$\mathcal{A} = (\mathbb{N}; \mathbb{N}; ; =). \quad f(x_1, \ldots, x_n) := \begin{cases} 1 & \text{if } x_i \neq x_j \text{ for all } i, j \text{ with } i \neq j, \\ 0 & \text{otherwise.} \end{cases}$$

Example (Further operators)

$$\underbrace{\ell : \; Z_j := \nu[f](\overset{z_1}{\overset{\downarrow}{Z_1}}, \ldots, \overset{z_n}{\overset{\downarrow}{Z_{I_1}}})}_{(1)}, \qquad \underbrace{\ell : \; Z_j := \nu_m[f](\overset{z_1}{\overset{\downarrow}{Z_1}}, \ldots, \overset{z_m}{\overset{\downarrow}{Z_m}})}_{(2)}$$

(1) We get a $z \in \mathbb{N} \setminus \{z_1, \ldots, z_n\}$ for any n and $(z_1, \ldots, z_n)$ with $z_i \neq z_{i+k}$.
(2) We get a $z \in \mathbb{N} \setminus \{z_1, \ldots, z_m\}$ for $(z_1, \ldots, z_m) \in U^m_{\mathcal{A}}$ with $z_i \neq z_{i+k}$. $(k \geq 1)$

Properties

(1) For the (ν-computable) correspondence $(z_1, \ldots, z_n) \mapsto \mathbb{N} \setminus \{z_1, \ldots, z_n\}$ we have not a computable choice function.
(2) For the correspondence $(z_1, \ldots, z_m) \mapsto \mathbb{N} \setminus \{z_1, \ldots, z_m\}$ we have a choice function which can be computed by means of $m + 1$ constants.

$\nu_{\min}$-Oracle BSS RAM's versus Simple BSS RAM's
(Motivation for Deterministic Operator $\nu_{\min}$)

$\mathcal{A} = (\mathbb{N} \times \mathbb{N}; \mathbb{N} \times \{0\}; s; \leq_{\text{lexi}})$, $\quad s(n, m) = (n, m + 1)$.

$\Rightarrow U_{\mathcal{A}}$ is not enumerable. $\quad \leq_{\text{lexi}}$ is a decidable well-ordering on $U_{\mathcal{A}}$.

Example (A deterministic operator)

$\nu_{\min}[f](x_1, \ldots, x_n) =_{\text{df}} \min \nu[f](x_1, \ldots, x_n)$

$\Rightarrow$ DETERMINISTIC! $\qquad \ell: \; Z_j := \nu_{\min}[f](Z_1, \ldots, Z_{I_1})$

$$f((n_1, m_1), (n_2, m_2)) =_{\text{df}} \begin{cases} a =_{\text{df}} (1, 0) & \text{if } n_1 = n_2, \\ \uparrow & \text{otherwise.} \end{cases}$$

$$g((n, m)) =_{\text{df}} \min\{(n', m') \mid f((n, m), (n', m')) = a\} = (n, 0)$$

Properties

f is computable by a BSS RAM over $\mathcal{A}$.
g is not computable by a BSS RAM over $\mathcal{A}$,
but it is computable by a $\nu_{\min}$-oracle BSS RAM over $\mathcal{A}$.
$\mathbb{N} \times \{0\}$ is not semi-decidable by a BSS RAM over $\mathcal{A}$,
but it is $\nu_{\min}$-semi-decidable by a $\nu_{\min}$-oracle BSS RAM over $\mathcal{A}$.

Definition (σ-Algebra)

$\mathfrak{A} \subseteq \mathcal{P}(\Omega)$ is a σ-*algebra* over Ω if

- $\Omega \in \mathfrak{A}$,
- $A, B \in \mathfrak{A} \Rightarrow A^c, A \cap B \in \mathfrak{A}$,
- $\bigcup_{i=0}^{\infty} A_i \in \mathfrak{A}$ for any sequence $A_0, A_1, \ldots \in \mathfrak{A}$.

Definition (Topology)

$\mathcal{T} \subseteq \mathcal{P}(\Omega)$ is a *topology* on Ω if

- $\Omega, \emptyset \in \mathcal{T}$,
- $A, B \in \mathcal{T} \Rightarrow A \cap B \in \mathcal{T}$,
- $\bigcup_{i \in I} A_i \in \mathcal{T}$ for any family $(A_i)_{i \in I}$ with $A_i \in \mathcal{T}$.

Definition (Borel sets)

The σ-Algebra $\sigma(\mathcal{T})$ generated by $\mathcal{T}$ is a σ-Algebra of *Borel sets*.

"The Arithmetical Hierarchy" in the BSS model

Computation over $\mathbb{R} = (\mathbb{R}; \mathbb{R}; \cdot, +, -; \leq)$ and Complete Problems (P. Cucker)

$$
\begin{array}{ccccccccc}
 & & & & \vdots & & & & \\
 & & \Sigma_3^{ND} & \nearrow\nwarrow & & \nwarrow\nearrow & \Pi_3^{ND} & & \\
 & & & & \Delta_3^{ND} & & & & \\
\text{Suslin's proj. hier.} & = & \Sigma_2^{ND} & \nearrow\nwarrow & & \nwarrow\nearrow & \Pi_2^{ND} & \ni & \mathrm{TOTAL}_{\mathbb{R}}, \mathrm{TOTAL}_{\mathbb{R}}^{ND} \\
\text{set of Borel sets} & = & & & \Delta_2^{ND} & & & & \\
(\subseteq \mathbb{R}^\infty) & & & \nearrow & & \nwarrow & & & \\
 & & & & \vdots & & & & \\
\mathrm{FIN}_{\mathbb{R}} & \in & \Sigma_2^0 & \nearrow\nwarrow & & \nwarrow\nearrow & \Pi_2^0 & & \\
 & & & & \Delta_2^0 & & & & \\
\Sigma_1^{ND} & = & \Sigma_1^0 & \nearrow\nwarrow & & \nwarrow\nearrow & \Pi_1^0 & = & \Pi_1^{ND} \ni \mathrm{INJ}_{\mathbb{R}} \\
 & & & & \Delta_1^0 & & & &
\end{array}
$$

Example (Complete problems, cf. P. Cucker)

$$
\begin{aligned}
\mathrm{FIN}_{\mathbb{R}} &= \{\mathrm{code}(\mathcal{M}) \mid (\exists n \in \mathbb{N})(\forall \vec{x} \in \mathbb{R}^{(\geq n)})(\mathcal{M}(\vec{x}) \uparrow)\} \\
\mathrm{INJ}_{\mathbb{R}} &= \{\mathrm{code}(\mathcal{M}) \mid (\forall \vec{x}_1, \vec{x}_2 \in \mathbb{R}^\infty)(\mathcal{M}(\vec{x}_1)\downarrow = \mathcal{M}(\vec{x}_2)\downarrow \Rightarrow \vec{x}_1 = \vec{x}_2)\} \\
\mathrm{TOTAL}_{\mathbb{R}}^{[ND]} &= \{\mathrm{code}(\mathcal{M}) \mid \mathcal{M} \in \mathsf{M}_{\mathbb{R}}^{[ND]} \;\&\; (\forall \vec{x} \in \mathbb{R}^\infty)(\mathcal{M}(\vec{x}) \downarrow)\}
\end{aligned}
$$

Definition (Measure)

$\mathfrak{A}$ *σ-algebra,* $\emptyset \in \mathfrak{A}$, $\mu : \mathfrak{A} \to [0, \infty]$.
μ *is a* measure *if* $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ *for pairw. disjoint* $A_i \in \mathfrak{A}$.

Example

Ω uncountable,
$\mathfrak{A} = \{A \subseteq \Omega \mid A$ or A^c is countable$\}$,

$$\mu_{\text{count}}(A) =_{\text{df}} \begin{cases} 0 & \text{if } A \in \mathfrak{A} \ \ \& \ \ A \text{ is countable,} \\ 1 & \text{if } A \in \mathfrak{A} \ \ \& \ \ A^c \text{ is countable.} \end{cases}$$

Example

Ω infinite,
$\mathfrak{A} = \{A \subseteq \Omega \mid A$ or A^c is finite$\}$,

$$\mu_{\text{fin}}(A) =_{\text{df}} \begin{cases} 0 & \text{if } A \in \mathfrak{A} \ \ \& \ \ A \text{ is finite,} \\ 1 & \text{if } A \in \mathfrak{A} \ \ \& \ \ A^c \text{ is finite.} \end{cases}$$

Definition (Measure)

$\mathfrak{A}$ *σ-algebra,* $\emptyset \in \mathfrak{A}$, $\mu : \mathfrak{A} \to [0, \infty]$.
μ *is a* measure *if* $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ *for pairw. disjoint* $A_i \in \mathfrak{A}$.

Example (Counting measure)

$$\mu_{\text{numb}}(A) =_{\text{df}} \begin{cases} |A| & \text{if } A \subseteq \Omega \ \& \ |A| < \infty, \\ \infty & \text{if } A \subseteq \Omega \ \& \ |A| = \infty. \end{cases}$$

Example (Dirac measure)

$$\delta_p(A) =_{\text{df}} \begin{cases} 1 & \text{if } p \in A, \\ 0 & \text{otherwise} \end{cases} \qquad (p \in \Omega).$$

Example

Lebesgue measure, Borel measure, Hausdorff measure

Measure Operators for BSS RAM's if $U_{\mathcal{A}} \supseteq \{1, 0\}$

$(\nu_{\delta_p}, \nu_{\text{count}}, \nu_{\text{fin}}, \ldots)$

$\mathcal{A}$ is fixed. $\{1, 0\} \subseteq U_{\mathcal{A}} = \Omega$.
$f : U_{\mathcal{A}}^{\infty} \to \{1, 0\}$ computable over $\mathcal{A}$.

$$\nu[f](x_1, \ldots, x_n) =$$
$$\{y_1 \in U_{\mathcal{A}} \mid (\exists (y_2, \ldots, y_m) \in U_{\mathcal{A}}^{\infty})(f(x_1, \ldots, x_n, y_1, y_2, \ldots, y_m) = 1)\}$$

Definition (Oracle instruction with measure operator ν_{δ_p})

$$\nu_{\delta_p}[f](x_1, \ldots, x_n) =_{\text{df}} \delta_p(\nu[f](x_1, \ldots, x_n))$$

DETERMINISTIC! $\ell : \; Z_j := \nu_{\delta_p}[f](Z_1, \ldots, Z_{I_1})$

Definition (Oracle instruction with measure operator ν_{count})

Ω uncountable, $\mathfrak{A} = \{A \subseteq \Omega \mid A$ or A^c is countable$\}$.

$$\nu_{\text{count}}[f](x_1, \ldots, x_n) =_{\text{df}} \begin{cases} \mu_{\text{count}}(A), & \text{if } A = \nu[f](x_1, \ldots, x_n) \in \mathfrak{A}, \\ \uparrow & \text{otherwise.} \end{cases}$$

DETERMINISTIC! $\ell : \; Z_j := \nu_{\text{count}}[f](Z_1, \ldots, Z_{I_1})$

Measure Operators for BSS RAM's if $U_{\mathcal{A}} \supseteq \mathbb{N}, \mathbb{R}^+$

$(\nu_{\text{numb}}, \nu^{[k]}_{\text{Lebes}})$

$\mathcal{A}$ is fixed, $\vec{x}, \vec{y} \in U^\infty_{\mathcal{A}}$, $f : U^\infty_{\mathcal{A}} \to \{1, 0\}$ computable over $\mathcal{A}$.

$\vec{\nu}[f](\vec{x}) =_{\text{df}} \{\vec{y} \in U^\infty_{\mathcal{A}} \mid f(\vec{x}\,.\,\vec{y}) = 1\}$, $\quad \vec{\nu}^{[1]}[f](\vec{x}) = \nu[f](\vec{x})$

$\vec{\nu}^{[k]}[f](\vec{x}) =_{\text{df}} \{\vec{w} \in U^k_{\mathcal{A}} \mid (\exists \vec{y} \in U^\infty_{\mathcal{A}})(f(\vec{x}\,.\,\vec{w}\,.\,\vec{y}) = 1)\}$ (Projection on $U^k_{\mathcal{A}}$)

Definition (Oracle instruction with measure operator $\vec{\nu}_{\text{numb}}$)

$\mathfrak{A} = \mathcal{P}(U^\infty_{\mathcal{A}})$, $\mathbb{N} \subseteq U_{\mathcal{A}}$.

$$\vec{\nu}_{\text{numb}}[f](\vec{x}) =_{\text{df}} \begin{cases} |A| & \text{if } A = \vec{\nu}[f](\vec{x}) \text{ and } |A| < \infty, \\ \uparrow & \text{otherwise.} \end{cases}$$

DETERMINISTIC! $\quad \ell : Z_j := \vec{\nu}_{\text{numb}}[f](Z_1, \ldots, Z_{I_1})$

Definition (Oracle instruction with measure operator $\vec{\nu}^{[k]}_{\text{Lebes}}$)

$\mathfrak{A} = \mathcal{P}(U^k_{\mathcal{A}})$, $\mathbb{R}^+ = \mathbb{R} \cap [0, \infty[\subseteq U_{\mathcal{A}}$.

$$\vec{\nu}^{[k]}_{\text{Lebes}}[f](\vec{x}) =_{\text{df}} \lambda^k(\nu^{[k]}[f](\vec{x}))$$

DETERMINISTIC! $\quad \ell : Z_j := \vec{\nu}^{[k]}_{\text{Lebes}}[f](Z_1, \ldots, Z_{I_1})$

$\mathcal{A}$ is fixed. $\mathbb{N} \subseteq U_{\mathcal{A}} = \Omega$, $p \in \mathbb{N}$. $f : U_{\mathcal{A}}^{\infty} \to \{1,0\}$ computable.

$\vec{\nu}[f](\vec{x}) = \{\vec{y} \in U_{\mathcal{A}}^{\infty} \mid f(\vec{x}\,.\,\vec{y}) = 1\}$

$M_{\vec{x}} =_{\text{df}} \{m \mid (\exists \vec{y} \in U_{\mathcal{A}}^{m})(\vec{y} \in \vec{\nu}[f](\vec{x}))\}$

Definition (Oracle instruction with measure operators for dim.)

$\{1,0\} \subseteq U_{\mathcal{A}}$ and $\mathbb{N} \subseteq U_{\mathcal{A}}$, resp.

$\nu^{\dim}_{\delta_p}[f](\vec{x}) =_{\text{df}} \nu_{\delta_p}(M_{\vec{x}})$

$\nu^{\dim}_{\text{fin}}[f](\vec{x}) =_{\text{df}} \nu_{\text{fin}}(M_{\vec{x}})$

$\nu^{\dim}_{\text{count}}[f](\vec{x}) =_{\text{df}} \nu_{\text{count}}(M_{\vec{x}})$

$\nu^{\dim}_{\text{numb}}[f](\vec{x}) =_{\text{df}} \nu_{\text{numb}}(M_{\vec{x}})$

DETERMINISTIC! $\ell :\ Z_j := \nu^{\dim}_{\ldots}[f](Z_1, \ldots, Z_{I_1})$

For $\mathcal{A}$:

- a finite number of operations & relations, all elements are constants,
- contains an infinite set effectively enumerable over $\mathcal{A}$: $\mathbb{N} \subseteq U_{\mathcal{A}}$.

$$\vdots$$

$$\mathrm{FIN}_{\mathbb{N}} \in \Sigma_2^0 \quad \nearrow\!\!\nwarrow \quad \nwarrow\!\!\nearrow \quad \Pi_2^0 \ni \mathrm{TOTAL}_{\mathbb{N}}, \mathrm{INCL}_{\mathbb{N}}$$

$$\Delta_2^0$$

$$\mathbb{H}_{\mathcal{A}}^{\mathrm{spec}}, \mathbb{H}_{\mathcal{A}} \in \Sigma_1^0 \quad \nearrow\!\!\nwarrow \quad \nwarrow\!\!\nearrow \quad \Pi_1^0$$

$$\Delta_1^0$$

Example (Complete problems, cf. C. Gaßner)

$\mathrm{FIN}_{\mathbb{N}}$	$=$	$\{\mathrm{code}(\mathcal{M}) \in U_{\mathcal{A}}^{\infty} \mid \lvert H_{\mathcal{M}} \cap \mathbb{N}^{\infty} \rvert < \infty\}$ ($H_{\mathcal{M}}$ = halting set)
$\mathrm{TOTAL}_{\mathbb{N}}$	$=$	$\{\mathrm{code}(\mathcal{M}) \in U_{\mathcal{A}}^{\infty} \mid (\forall \vec{x} \in \mathbb{N}^{\infty})(\mathcal{M}(\vec{x}) \downarrow)\}$
$\mathrm{INCL}_{\mathbb{N}}$	$=$	$\{(\mathrm{code}(\mathcal{M}) \,.\, \mathrm{code}(\mathcal{N})) \in U_{\mathcal{A}}^{\infty} \mid (H_{\mathcal{M}} \cap \mathbb{N}^{\infty}) \subseteq (H_{\mathcal{N}} \cap \mathbb{N}^{\infty})\}$
$\mathbb{H}_{\mathcal{A}}^{[\mathrm{spec}]}$	$\widehat{=}$	Halting problems for BSS RAM's over $\mathcal{A}$

Decision of the Halting Problem by Measure Oracles

$\mathbb{H}_{\mathcal{A}} - \nu_{\delta_1}, \mathbb{H}_{\mathcal{A}} - \nu_{\text{numb}}, \ldots$

$\mathbb{H}_{\mathcal{A}} = \{(\vec{x} \,.\, \text{code}(\mathcal{M})) \mid \mathcal{M}(\vec{x}) \downarrow\}$, finite many operations & relations,
$\mathbb{N} \subseteq U_{\mathcal{A}}$, $\{\text{code}(\mathcal{M}) \mid \mathcal{M}$ is machine$\}$ prefix-free and decidable.

Example (Decidable halting problems)

$\mathbb{H}_{\mathcal{A}}^{(1)} = \{((x_1, 0, \ldots, x_{n-1}, 0, x_n, 1) \,.\, \text{code}(\mathcal{M}) \,.\, 1 \,.\, k) \mid \mathcal{M}(\vec{x}) \downarrow^{\leq k}\}$ is decidable.
$\mathbb{H}_{\mathcal{A}}^{(2)} = \{((x_1, 0, \ldots, x_{n-1}, 0, x_n, 1) \,.\, \text{code}(\mathcal{M}) \,.\, k) \mid \mathcal{M}(\vec{x}) \downarrow^{=k}\}$ is decidable.
$\mathbb{H}_{\mathcal{A}}^{(3)} = \{(\underbrace{(x_1, 0, \ldots, x_{n-1}, 0, x_n, 1) \,.\, \text{code}(\mathcal{M})}_{\langle \vec{x} \,.\, \text{code}(\mathcal{M}) \rangle} \,.\, k) \mid \mathcal{M}(\vec{x}) \downarrow^{\leq k}\}$ is decidable.

Example (Decision of $\mathbb{H}_{\mathcal{A}}$)

Input: $(\vec{x} \,.\, \text{code}(\mathcal{M}))$; Output: $\nu_{\delta_1}[\chi_{\mathbb{H}_{\mathcal{A}}^{(1)}}](\langle \vec{x} \,.\, \text{code}(\mathcal{M}) \rangle)$.
Input: $(\vec{x} \,.\, \text{code}(\mathcal{M}))$; Output: $\nu_{\text{numb}}[\chi_{\mathbb{H}_{\mathcal{A}}^{(2)}}](\langle \vec{x} \,.\, \text{code}(\mathcal{M}) \rangle)$.
Input: $(\vec{x} \,.\, \text{code}(\mathcal{M}))$; Output: $\nu_{\text{numb}}^{\text{dim}}[\chi_{\mathbb{H}_{\mathcal{A}}^{(3)}}(\langle \vec{x} \,.\, \text{code}(\mathcal{M}) \rangle)$.

Example (Decision of $\mathbb{H}_{\mathcal{A}}$ for $U_{\mathcal{A}} = \mathbb{N}$)

Input: $(\vec{x} \,.\, \text{code}(\mathcal{M}))$; Output: $\nu_{\text{fin}}[\chi_{\mathbb{H}_{\mathcal{A}}^{(3)}}](\langle \vec{x} \,.\, \text{code}(\mathcal{M}) \rangle)$.

$\mathcal{A} = (\mathbb{N}; \mathbb{N}; s, \ldots; =, \ldots)$, finite many operations and relations,
$s(n) = n + 1$.
$\mathcal{M}_f$ computes f.

Example (Semi-decision whether $y \in \nu[f](\vec{x})$)

$\mathcal{N}_{f,\vec{x}}$:

Input y;
for $k = 1, 2, \ldots$
simulate the first k steps of $\mathcal{M}_f$ on

$$\{x_1\} \times \cdots \times \{x_n\} \times \{y\} \times \underbrace{\{0, \ldots, k\} \times \cdots \times \{0, \ldots, k\}}_{\in \mathbb{N}^l \quad (l \leq k)}$$

until
an output of $\mathcal{M}_f$ is reached and this output is 1. (Then halt.)

Example (Computation of $\nu_{\delta_p}[f](\vec{x})$)

Input: $\vec{x}$; Output: $\chi_{\mathbb{H}_{\mathcal{A}} \cap \{(p \,.\, \mathrm{code}(\mathcal{N}_{f,\vec{x}}))\}}$.

$\mathcal{A} = (\mathbb{N}; \mathbb{N}; s, \ldots; =, \ldots)$, finite many operations and relations,
$s(n) = n + 1$. $\mathcal{M}_f$ computes f.

Example (Semi-decision whether $(y+m) \in \nu[f](\vec{x})$)

$\mathcal{N}^m_{f,\vec{x}}$:

Input y;
for $k = 1, 2, \ldots$
simulate the first k steps of $\mathcal{M}_f$ on

$$\{x_1\} \times \cdots \times \{x_n\} \times \{y+m\} \times \underbrace{\{0,\ldots,k\} \times \cdots \times \{0,\ldots,k\}}_{\in \mathbb{N}^l \quad (l \leq k)}$$

until
an output of $\mathcal{M}_f$ is reached and this output is 1. (Then halt.)

Example (Computation of $\nu_{\mathrm{fin}}[f](\vec{x})$)

Input: $\vec{x}$;
if $\mathrm{code}(\mathcal{N}^0_{f,\vec{x}}) \in \mathrm{FIN}_{\mathbb{N}}$ then Output: 0;
else for $m = 1, 2, \ldots$ do if $\mathrm{code}(\mathcal{N}^m_{f,\vec{x}}) \in \mathrm{TOTAL}_{\mathbb{N}}$ then Output: 1.

Computation of Measure Operators by $\mathrm{FIN}_{\mathbb{N}}$ & ...

$\mathcal{A} = (\mathbb{N};\mathbb{N}; s,\ldots;=,\ldots)$, finite many operations and relations, $s(n) = n+1$. $\mathcal{M}_f$ computes f.

Example (Useful for deciding whether $(\mathbb{N}+m)\cap\nu[f](\vec{x}) = \emptyset$)

$\mathcal{K}^m_{f,\vec{x}}$:

Input (y,k);
simulate the first k steps of $\mathcal{M}_f$ on

$$\{x_1\}\times\cdots\times\{x_n\}\times\{y+m\}\times\underbrace{\{0,\ldots,k\}\times\cdots\times\{0,\ldots,k\}}_{\in\mathbb{N}^l\quad(l\leq k)}$$

if the output of $\mathcal{M}_f$ is reached and the output is 1 then loop;
else halt.

Example (Computation of $\nu_{\mathrm{numb}}[f](\vec{x})$)

Input: $(x_1,\ldots,x_n)$
if $\mathrm{code}(\mathcal{N}_{f,\vec{x}})\in\mathrm{FIN}_{\mathbb{N}}$ then
for $m = 0,1,2,\ldots$ do if $\mathrm{code}(\mathcal{K}^m_{f,\vec{x}})\in\mathrm{TOTAL}_{\mathbb{N}}$ then
$\{s := 0$; for $w = 0,1,\ldots,m$ do if $\mathcal{N}_{f,\vec{x}}(w)\downarrow$ then s:= s+1; Output: s.$\}$

Decision of Solubility of Systems of Equations

by Means of ν_{numb} and $\nu_{\text{Lebes}_{[0,1]}}$, ν_{count} or ν_{δ_p} in Constant Time

$$\mathcal{A} = (\mathbb{R};\mathbb{R};+,\varphi_{c_1},\dots,\varphi_{c_k};=) \qquad (c_i \in \mathbb{R},\ \varphi_c(x) =_{\text{df}} cx).$$

$$f(\vec{w}\,.\,\vec{x}) = \begin{cases} 1 & \text{if } \vec{w} = (n,m,a_{1,1},\dots,a_{n,m},b_1,\dots,b_m) \\ & \quad \& \ \vec{x} = (x_1,\dots,x_n)\ \&\ (\forall j \le m)(\sum_{i=1}^{n} a_{i,j}x_i = b_j), \\ 0 & \text{otherwise.} \end{cases}$$

$\Rightarrow f$ computable over $\mathcal{A}$

$$\text{LinEq} = \{\vec{w} \mid (\exists \vec{x})(f(\vec{w}\,.\,\vec{x}) = 1)\} \notin \text{P}_{\mathcal{A}}$$

(cf. K. Meer, P. Koiran)

Example (Decision whether $\vec{w} \in \text{LinEq}$ with measure operators)

For input $\vec{w}$ three possibilities:

- if $\nu_{\delta_1}[f](\vec{w}) = 1$ then Output 1; else Output $\text{sgn}(\nu_{\text{numb}}[f](\vec{w}))$;

  yes $\hat{=}\ \pi_{nm+m+3}(f^{-1}(\{1\}))$ is $\{1\}$ or $\mathbb{R}$ no $\hat{=}\ \pi_{nm+m+3}(f^{-1}(\{1\})) = \{a\}$ $(a \neq 1)$

- if $\nu_{\text{count}}[f](\vec{w}) = 1$ then Output 1; else Output $\text{sgn}(\nu_{\text{numb}}[f](\vec{w}))$;

  yes $\hat{=}\ \mathbb{R} \setminus \pi_{nm+m+3}(f^{-1}(\{1\}))$ countable, thus empty no $\hat{=}\ \pi_{nm+m+3}(f^{-1}(\{1\})) = \{a\}$

- if $\nu_{\text{Lebes}_{[0,1]}}[f](\vec{w}) \neq 0$ then Output 1; else Output $\text{sgn}(\nu_{\text{numb}}[f](\vec{w}))$;

  yes $\hat{=}\ \pi_{nm+m+3}(f^{-1}(\{1\})) \cap [0,1] = [0,1]$ no $\hat{=}\ \pi_{nm+m+3}(f^{-1}(\{1\})) = \{a\}$

$$\mathcal{A} = (\mathbb{R}; \mathbb{R}; +, -; \leq)$$

$$f(\vec{w} . \vec{x}) = \begin{cases} 1 & \text{if } \vec{w} = (n, m, a_{1,1}, \ldots, a_{n,m}, b_1, \ldots, b_m) \\ & \quad \& \ \vec{x} = (x_1, \ldots, x_n) \ \& \ (\forall j \leq m)(\sum_{i=1}^{n} a_{i,j} x_i \leq b_j), \\ 0 & \text{otherwise.} \end{cases}$$

$\Rightarrow f$ computable over $\mathcal{A}$

$$\text{LinIneq} = \{\vec{w} \mid (\exists \vec{x})(f(\vec{w} . \vec{x}) = 1)\}$$

Example (Semi-decision whether $\vec{w} \in \text{LinIneq}$ with measure operators)

Input: $\vec{w}$;
for $s := 1, 2, \ldots$ do
if $\nu_{\text{Lebes}_{[-s,s]}}[f](\vec{w}) \neq 0$ then Output 1; "yes" $\hat{=} \pi_{nm+m+3}(f^{-1}(\{1\})) \cap [-s, s] \neq \emptyset$
else if $\text{sgn}(\nu_{\text{numb}}[f_{|\{\vec{y} \in \mathbb{R}^\infty | y_{nm+m+3} \in [-s,s]\}}](\vec{w})) = 1$ then Output 1;

"yes" $\hat{=} \pi_{nm+m+3}(f^{-1}(\{1\})) \cap [-s, s] = \{a\}$

$\mathcal{A} = (\mathbb{R}; \mathbb{R}; +, -; \leq)$

$$f(\vec{w}\,.\,\vec{x}) = \begin{cases} 1 & \text{if } \vec{w} = (n, m, a_{1,1}, \ldots, a_{n,m}, b_1, \ldots, b_m) \\ & \quad \& \ \vec{x} = (x_1, \ldots, x_n) \ \& \ (\forall j \leq m)(\sum_{i=1}^{n} a_{i,j} x_i \leq b_j), \\ 0 & \text{otherwise.} \end{cases}$$

$\Rightarrow f$ computable over $\mathcal{A}$

$\text{LinIneq} = \{\vec{w} \mid (\exists \vec{x} \in \mathbb{R}^\infty)(f(\vec{w}\,.\,\vec{x}) = 1)\} \in \mathrm{P}_{\mathcal{A}}$?

$\text{LinIneq} \in \mathrm{DNP}_{\mathcal{A}}$ (cf. P. Koiran)

$\Rightarrow$ There are computable g and p with

$\text{LinIneq} = \{\vec{w} \mid (\exists \vec{x} \in \{0,1\}^{p(|\vec{w}|)})(g(\vec{w}\,.\,\vec{x}) = 1)\}$

Example (Decision whether $\vec{w} \in \text{LinIneq}$ with measure operators)

Input: $\vec{w}$; if $\nu_{\delta_1}[g](\vec{w}) = 1$ or $\nu_{\delta_0}[g](\vec{w}) = 1$ then Output 1;
else Output 0;

no $\hat{=}\ \pi_{nm+m+3}(g^{-1}(\{1\})) = \emptyset$

Input: $\vec{w}$; Output $\mathrm{sgn}(\nu_{\text{numb}}[g](\vec{w}))$;

since $\pi_{nm+m+3}(g^{-1}(\{0,1\})) \subseteq \{0,1\}$

Thank you very much for your attention!

References

L. BLUM, M. SHUB, and S. SMALE: "On a theory of computation and complexity over the real numbers: NP-completeness, recursive functions and universal machines" (1989)

F. CUCKER: "The arithmetical hierarchy over the reals" (1992)

C. GASSNER: "Computation over algebraic structures and a classification of undecidable problems" (2016)

S. C. KLEENE: "Introduction to metamathematics" (1952).

Y. N. MOSCHOVAKIS: "Abstract first order computability. I" (1969)