Effective Descriptions of Mathematical Objects and the BSS-RAM Model

Christine Gaßner

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$\Downarrow$ Stephen C. Kleene & Yiannis N. Moschovakis & Klaus Weihrauch etc.
Recursion Theory based on recursion and μ-operator or ν-operator
Type-2 Turing machines

$\Downarrow$ Gaßner
ν-Operator, ν_{lim}-Operator, and Type-2 BSS-RAM's over structures $\mathcal{A}$

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Computable multi-valued correspondences and limits over$\mathcal{A}$

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Questions

- Which machines can simulate Type-2 Turing machines?
- Is it possible to compute representations?

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Outline

- BSS RAM's
- Type-2 BSS-RAM's
- The simulation of Type-2 Turing machines
- Representations for the open semi-decidable sets

Computation over $\mathcal{A} = (\underbrace{U_{\mathcal{A}}}_{\text{universe}}; \underbrace{C_{\mathcal{A}}}_{\text{constants}}; \underbrace{f_1, \ldots, f_{n_1}}_{\text{operations}}; \underbrace{R_1, \ldots, R_{n_2}, =}_{\text{relations}})$.

Z_1	Z_2	Z_3	Z_4	Z_5	$\ldots$

Registers for elements in $U_{\mathcal{A}}$

I_1	I_2	I_3	I_4	$\ldots$	$I_{k_{\mathcal{M}}}$

Registers for indices in $\mathbb{N}$

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Registers for indices in $\mathbb{N}$

- Computation instructions:

  $\ell : \; Z_j := f_k(Z_{j_1}, \ldots, Z_{j_{m_k}})$ (e. g. $\ell : \; Z_j := Z_{j_1} + Z_{j_2}$)

  $\ell : \; Z_j := d_k$ $(d_k \in C_{\mathcal{A}} \subseteq U_{\mathcal{A}})$
- Branching instructions:

  $\ell : \;$ if $Z_j = Z_k$ then goto ℓ_1 else goto ℓ_2

  $\ell : \;$ if $R_k(Z_{j_1}, \ldots, Z_{j_{n_k}})$ then goto ℓ_1 else goto ℓ_2
- Copy instructions:

  $\ell : \; Z_{I_j} := Z_{I_k}$
- Index instructions:

  $\ell : \; I_j := 1$

  $\ell : \; I_j := I_j + 1$

  $\ell : \;$ if $I_j = I_k$ then goto ℓ_1 else goto ℓ_2

- $U_{\mathcal{A}}$ is the universe of $\mathcal{A}$
- Input and output space: $U_{\mathcal{A}}^{\infty} =_{\mathrm{df}} \bigcup_{i\geq 1} U_{\mathcal{A}}^{i}$

- Input of $\vec{x} = (x_1, \ldots, x_n) \in U_{\mathcal{A}}^{\infty}$:

$x_1 \quad x_2 \quad x_3 \quad x_4 \qquad\qquad x_n$

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x_1	x_2	x_3	x_4		x_n	x_n	x_n	
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$		$\downarrow$	$\downarrow$	$\downarrow$	
Z_1	Z_2	Z_3	Z_4	$\ldots$	Z_n	Z_{n+1}	Z_{n+2}	$\ldots$

Uniform Computation over Algebraic Structures
(Input and Output Procedures of Machines in $\mathrm{M}_{\mathcal{A}}$)

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$$
\begin{array}{|c|c|c|c|c|c|c|c|c}
x_1 & x_2 & x_3 & x_4 & & x_n & x_n & x_n & \\
\downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & \\
\hline
Z_1 & Z_2 & Z_3 & Z_4 & \ldots & Z_n & Z_{n+1} & Z_{n+2} & \ldots \\
\hline
\end{array}
$$

$$
\begin{array}{|c|c|c|c|c|}
\hline
I_1 & I_2 & I_3 & \ldots & I_{k_{\mathcal{M}}} \\
\hline
\uparrow & \uparrow & \uparrow & & \uparrow \\
n & 1 & 1 & & 1
\end{array}
$$

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$$\begin{array}{ccccccccc} x_1 & x_2 & x_3 & x_4 & & x_n & x_n & x_n & \\ \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow & \\ \hline \multicolumn{1}{|c|}{Z_1} & \multicolumn{1}{c|}{Z_2} & \multicolumn{1}{c|}{Z_3} & \multicolumn{1}{c|}{Z_4} & \multicolumn{1}{c|}{\ldots} & \multicolumn{1}{c|}{Z_n} & \multicolumn{1}{c|}{Z_{n+1}} & \multicolumn{1}{c|}{Z_{n+2}} & \ldots \\ \hline \end{array}$$

$$\begin{array}{|c|c|c|c|c|} \hline I_1 & I_2 & I_3 & \ldots & I_{k_{\mathcal{M}}} \\ \hline \multicolumn{1}{c}{\uparrow} & \multicolumn{1}{c}{\uparrow} & \multicolumn{1}{c}{\uparrow} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{\uparrow} \\ \multicolumn{1}{c}{n} & \multicolumn{1}{c}{1} & \multicolumn{1}{c}{1} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{1} \end{array}$$

$\nearrow$

!!!

Uniform Computation over Algebraic Structures

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Z_1	Z_2	Z_3	Z_4	$\ldots$	Z_n	Z_{n+1}	Z_{n+2}	$\ldots$

I_1	I_2	I_3	$\ldots$	$I_{k_{\mathcal{M}}}$
$\uparrow$	$\uparrow$	$\uparrow$		$\uparrow$
n	1	1		1

$\nearrow$

!!!

- Output of $Z_1, \ldots, Z_{I_1}$.

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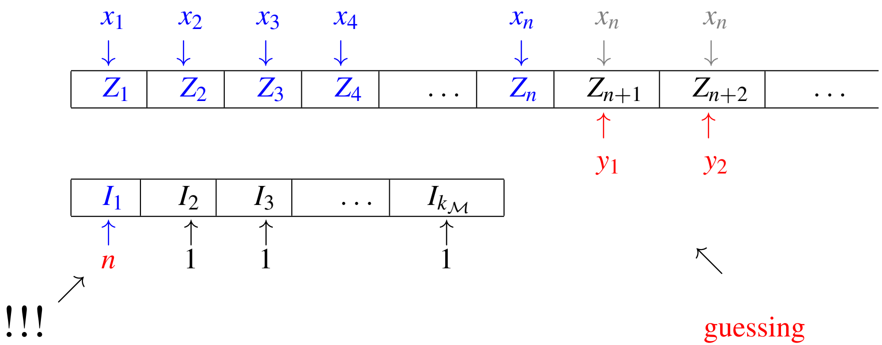

- Output of $Z_1, \ldots, Z_{I_1}$.

Functions Computable over $\mathcal{A}$

(Deterministic Computation and Nondeterministic Computation)

$f :\subseteq U_{\mathcal{A}}^{\infty} \to U_{\mathcal{A}}^{\infty}$ *partially defined function*

$f :\subseteq U_{\mathcal{A}}^{\infty} \to \mathcal{P}(U_{\mathcal{A}}^{\infty})$

$f :\subseteq U_{\mathcal{A}}^{\infty} \rightrightarrows U_{\mathcal{A}}^{\infty}$ *multiple-valued function*

f *nondet.* / μ- / ν- / ν_{lim}- *computable* if it is computable by nondet. / μ-oracle / ν-oracle / ν_{lim}-oracle BSS RAM.

$\mathcal{A}$ fixed, $\mathbb{N} \subseteq U_{\mathcal{A}}$ effectively enumerable over $\mathcal{A}$,

μ-Oracle BSS-RAM's with μ-Operator for $\mathbb{N} \subseteq U_{\mathcal{A}}$
(Kleene's Operator μ)

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Definition (Kleene's operator for $\mathcal{A}$)

$\mu[f](x_1, \ldots, x_n)$
$=_{\mathrm{df}} \min\{k \in \mathbb{N} \mid f(x_1, \ldots, x_n, k) = 1 \;\&\; f(x_1, \ldots, x_n, l)\!\downarrow$ for $l < k, l \in \mathbb{N}\}$

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Example

$\mathcal{A} = (\mathbb{N}; 0; +, -; \leq, =)$

$$f_0(a_1, \ldots, a_n, x) := \begin{cases} 1 & \text{if } \underbrace{x^n + a_n x^{n-1} + \cdots + a_1 x^0}_{p(x)} = 0, \\ 0 & \text{otherwise.} \end{cases}$$

$\Rightarrow \mu[f_0](a_1, \ldots, a_n) =$ the smallest zero of p

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Definition (Oracle instruction with Kleene's operator)

$$\ell : \; Z_j := \mu[f](\underset{\overset{\downarrow}{z_1}}{Z_1},\ldots,\underset{\overset{\downarrow}{z_n}}{Z_{I_1}}) \qquad (\text{if } I_1 = n)$$

no minimum $\Rightarrow$ the machine loops forever

μ-Oracle BSS-RAM's with μ-Operator for $\mathbb{N} \subseteq U_{\mathcal{A}}$

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no minimum $\Rightarrow$ the machine loops forever

Properties

Any μ-semi-decidable problem is semi-decidable over $\mathcal{A}$.

ν-Oracle BSS-RAM's for Structures with a and b
(Moschovakis' Operator ν)

$\mathcal{A}$ fixed, $\quad a, b$ constants of $\mathcal{A}$, $\quad f :\subseteq U_{\mathcal{A}}^{\infty} \to \{a, b\}$ computable over $\mathcal{A}$.

Definition (Moschovakis' operator for $\mathcal{A}$)

$\nu[f](x_1, \ldots, x_n)$
$=_{\mathrm{df}} \{y_1 \in U_{\mathcal{A}} \mid (\exists (y_2, \ldots, y_m) \in U_{\mathcal{A}}^{\infty})(f(x_1, \ldots, x_n, y_1, y_2, \ldots, y_m) = a)\}$

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Properties (Guessing by ν-operator of a ν-oracle machine)

$$\begin{array}{ll} & \;\; x_1 \;\cdots\; x_n \\ & \;\;\downarrow \quad\;\;\; \downarrow \\ Z_j := \nu[f](Z_1, \ldots, Z_{I_1}); & \ldots \\ \downarrow & \\ y_1 & \end{array} \qquad \begin{array}{ll} & \;\; x_1 \;\cdots\; x_n \quad y_1 \\ & \;\;\downarrow \quad\;\;\; \downarrow \quad\;\; \downarrow \\ Z_j := \nu[f](Z_1, \ldots, Z_{I_1 - 1}, Z_{I_1}) & \ldots \\ \downarrow & \\ y_2 & \end{array}$$

NONDETERMINISTIC!

$$\Rightarrow f(x_1, \ldots, x_n, y_1, \ldots, y_m) = a$$

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$$\underset{\substack{\downarrow\\ y_1}}{Z_j} := \nu[f](\overset{\substack{x_1\\ \downarrow}}{Z_1}, \ldots, \overset{\substack{x_n\\ \downarrow}}{Z_{I_1}}); \ \ldots \ \underset{\substack{\downarrow\\ y_2}}{Z_j} := \nu[f](\overset{\substack{x_1\\ \downarrow}}{Z_1}, \ldots, \overset{\substack{x_n\\ \downarrow}}{Z_{I_1-1}}, \overset{\substack{y_1\\ \downarrow}}{Z_{I_1}}) \quad \ldots$$

NONDETERMINISTIC!

$$\Rightarrow f(x_1, \ldots, x_n, y_1, \ldots, y_m) = a$$

Proposition

$A \subseteq U_{\mathcal{A}}^{\infty}$ is ν-semi-decidable iff A is nondeterm. semi-decidable.

Definition (Operator $\nu_{\lim}$ for $\mathcal{A}$)

$$\nu_{\lim}[f](x_1,\ldots,x_n) =_{\mathrm{df}} \lim_{k\to\infty} u_k \quad \text{if } f^{-1}(\{a\}) = \bigcup_k \{(x_1,\ldots,x_n,u_k,a,\overbrace{b,\ldots,b}^{k \text{ times}})\}$$

Properties

$Z_j := \nu_{\lim}[f](Z_1,\ldots,Z_{I_1})$ DETERMINISTIC!

$\nu_{\lim}$-Oracle BSS-RAM's and Type-2 BSS-RAM's
(Operator $\nu_{\lim}$)

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Properties

$Z_j := \nu_{\lim}[f](Z_1,\ldots,Z_{I_1})$ DETERMINISTIC!

Proposition

g is $\nu_{\lim}$-computable if g is computable by a Type-2 BSS-RAM.

This Type-2 RAM works

- as a BSS-RAM without output procedure,
- with an infinite only-write output tape.

An infinite-limit Type-2 BSS-RAM provides

- only the limit (if it exists).

Simulation of Type-2 Turing Machines

(Computation of $f :\subseteq \mathbb{R} \rightrightarrows \mathbb{R}$ over $\mathcal{A} = (\mathbb{R}; 0, 1, \ldots; +, -, \cdot, :, \ldots; \leq, \ldots))$

$(\mathbb{R}, \delta_{\mathbb{R}})$	$\delta_{\mathbb{R}}(0^{n_0}10^{n_1}1\cdots) = x \in \mathbb{R}$ with $\lvert x - \nu_{\mathbb{Q}}(n_i)\rvert < \frac{1}{2^i}$	$(\nu_{\mathbb{Q}} : \mathbb{N} \to \mathbb{Q})$
f	$:\subseteq \mathbb{R} \rightrightarrows \mathbb{R}$ computable	
F	$:\subseteq \{0,1\}^\omega \to \{0,1\}^\omega$ a realizer of f	(with $\delta_{\mathbb{R}}(F(p)) \in f(\delta_{\mathbb{R}}(p))$)
M_F	Type-2 Turing machine computing F	(with $\mathrm{code}(M_F) \in \{0,1\}^\infty$)

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Program (Computing f by a nondet. $\mathcal{A}$-machine $\mathcal{M}_f$)

Input: $(r \,.\, \mathrm{code}(M_F)) \in \mathbb{R}^\infty$.

Guess: $0.\overbrace{0^{n_0}10^{n_1}1\cdots}^{p} \in \mathbb{R}$ with p as candidate for representing $r \in \mathbb{R}$

$k = 1;$

1: if $(\lvert r - \nu_{\mathbb{Q}}(n_i)\rvert < \frac{1}{2^i}$ *for* $i \leq k)$

then simulate M_F on $0^{n_0}10^{n_1}1\cdots 0^{n_k}1$;

store computed output values of M_F in $Z_2, Z_4, \ldots, Z_{2l_k}$;

if (*value 1 should be written on the output tape of* M_F)

then add the corresponding $\nu_{\mathbb{Q}}(n_{i_k})$ on the output tape;

if (*a new input value is necessary*) then $\{k := k+1;$ goto $1\}$

Output: the limit.

Descriptions for Open BSS-Semi-Decidable Sets
(Deterministic for $\mathcal{A} = (\mathbb{R}; \mathbb{R}; +, -, \cdot, :; \leq)$)

$\delta_o(p)$	$= A \in \mathscr{O}(\mathbb{R})$ if p describes $A = \bigcup_i [q_i, \bar{q}_i]$ $\quad (q_i, \bar{q}_i \in \mathbb{Q})$
$\mathcal{M}_A$	$\mathcal{A}$-machine computing $\chi_A :\subseteq \mathbb{R} \to \{0,1\}$
w	$= w_0 w_1 \cdots \in \{0,1\}^*$ describes a computation path
	$w_0 = 0 \Rightarrow$ only "$<$"-tests and "$>$"-tests
	$w_0 = 1 \Rightarrow$ at least one "$=$"-test
$B_{\mathcal{M}_A}(x)$	computation path of $\mathcal{M}_A$ on x
$h_1^w, \ldots, h_{s_w}^w$	the functions evaluated along w $\quad (h_i^w(x) = 0?, \ldots)$
$(*)_{(\mathcal{M}_A, w)}$	None of the zeros of h_i^w belongs to $[\frac{i}{2^l}, \frac{i+1}{2^l}]$.
$C_l^w(A)$	$= \bigcup_{i=-lb^l}^{lb^l-1} \bigcup_{\substack{x=\frac{2i+1}{2^{l+1}} \ \& \ w \text{ is an accept. path } \& \ (*)_{(\mathcal{M}_A,w)} \\ (B_{\mathcal{M}_A}(x)=w \text{ for } w_0=0) \ \& \ (B_{\mathcal{M}_A}(x)\neq w \text{ for } w_0=1)}} [\frac{i}{2^l}, \frac{i+1}{2^l}]$

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$$\Rightarrow A = \lim_{l\to\infty} \bigcup_{|0w| \leq l} C_l^{0w}(A) \cup \bigcup_{1w \in \{0,1\}^+} (\mathbb{R} \setminus \bigcup_{l\to\infty} C_l^{1w}(A))$$

Properties

$\mathrm{code}(\mathcal{M}_A) \mapsto \langle p_0, p_1, p_2, \ldots \rangle$ with $\delta_o(p_0) \cup \bigcup_{s\to\infty} (\mathbb{R} \setminus \delta_o(p_s)) = A$ is computable by a deterministic Type-2 BSS-RAM over $\mathcal{A}$.

Note: $\mathrm{bin}(s) \in \{1w \mid w \in \{0,1\}^*\}$, $s \geq 1$

Descriptions for Open BSS-Semi-Decidable Sets
(Nondeterministic for $\mathcal{A} = (\mathbb{R}; \mathbb{R}; +, -, \cdot, :; \leq)$)

$\delta_o(p) \quad = A \in \mathscr{O}(\mathbb{R})$ if p describes $A = \bigcup_i [q_i, \bar{q}_i] \qquad (q_i, \bar{q}_i \in \mathbb{Q})$

$\mathcal{M}_A \quad \mathcal{A}$-machine computing $\chi_A :\subseteq \mathbb{R} \to \{0, 1\}$

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$\mathcal{M}_A \quad$ $\mathcal{A}$-machine computing $\chi_A :\subseteq \mathbb{R} \to \{0,1\}$

Program (Computing $\mathrm{code}(\mathcal{M}_B) \mapsto p \in \delta_o^{-1}(B)$ by an $\mathcal{A}$-machine)

Input: $\mathrm{code}(\mathcal{M}_B) \in \mathbb{R}^\infty$.

Guess: $0.\overbrace{0^{n_0}1^{n_1}0^{n_2}\cdots}^{p} \in \mathbb{R}$ with p as a candidate for describing B.

$k := 1;$

if $(\mathrm{code}(\mathcal{M}_B), \mathrm{code}(\mathcal{M}_{\delta_o(p)})) \notin \mathrm{INCL}_{\mathcal{A}}^{\mathrm{ND}}$ then $k := 0;$

if $(\mathrm{code}(\mathcal{M}_{\delta_o(p)}), \mathrm{code}(\mathcal{M}_B)) \notin \mathrm{INCL}_{\mathcal{A}}^{\mathrm{ND}}$ then $k := 0;$

if $k = 1$ then

write p on the output tape.

$$\mathrm{INCL}_{\mathcal{A}}^{\mathrm{ND}} = \{(\mathrm{code}(\mathcal{M}).\,\mathrm{code}(\mathcal{N})) \mid (\mathcal{M},\mathcal{N}) \in \mathsf{M}_{\mathcal{A}} \times \mathsf{M}_{\mathcal{A}} \,\&\, H_{\mathcal{M}} \subseteq H_{\mathcal{N}}\}$$

Descriptions for Open BSS-Semi-Decidable Sets
(Nondeterministic for $\mathcal{A} = (\mathbb{R}; \mathbb{R}; +, -, \cdot, :; \leq)$)

$\delta_o(p)$ $= A \in \mathscr{O}(\mathbb{R})$ if p describes $A = \bigcup_i [q_i, \bar{q}_i]$ $(q_i, \bar{q}_i \in \mathbb{Q})$
$\mathcal{M}_A$ $\mathcal{A}$-machine computing $\chi_A :\subseteq \mathbb{R} \to \{0, 1\}$

Program (Computing $\mathrm{code}(\mathcal{M}_B) \mapsto p \in \delta_o^{-1}(B)$ by an $\mathcal{A}$-machine)

Input: $\mathrm{code}(\mathcal{M}_B) \in \mathbb{R}^\infty$.

Guess: $0.\overbrace{0^{n_0}1^{n_1}0^{n_2}\cdots}^{p} \in \mathbb{R}$ with p as a candidate for describing B.
$k := 1$;
if $(\mathrm{code}(\mathcal{M}_B), \mathrm{code}(\mathcal{M}_{\delta_o(p)})) \notin \mathrm{INCL}^{\mathrm{ND}}_{\mathcal{A}}$ then $k := 0$;
if $(\mathrm{code}(\mathcal{M}_{\delta_o(p)}), \mathrm{code}(\mathcal{M}_B)) \notin \mathrm{INCL}^{\mathrm{ND}}_{\mathcal{A}}$ then $k := 0$;
if $k = 1$ then
write p on the output tape.

$$\mathrm{INCL}^{\mathrm{ND}}_{\mathcal{A}} = \{(\mathrm{code}(\mathcal{M}).\,\mathrm{code}(\mathcal{N})) \mid (\mathcal{M}, \mathcal{N}) \in \mathsf{M}_{\mathcal{A}} \times \mathsf{M}_{\mathcal{A}} \,\&\, H_{\mathcal{M}} \subseteq H_{\mathcal{N}}\}$$

$\mathrm{INCL}^{\mathrm{ND}}_{\mathcal{A}}$ is $\mathcal{A}$-Π^{ND}_2-complete.

A Hierarchy over $\mathcal{A}$
(Analogously to the Arithmetical Hierarchy)

A finite number of operations and relations, all elements constants.

$$
\begin{aligned}
\Sigma_0^{\mathrm{ND}} &=_{\mathrm{df}} \mathrm{DEC}_{\mathcal{A}} \\
\Pi_n^{\mathrm{ND}} &=_{\mathrm{df}} \{U_{\mathcal{A}}^{\infty} \setminus P \mid P \in \Sigma_n^{\mathrm{ND}}\} \\
\Delta_n^{\mathrm{ND}} &=_{\mathrm{df}} \Sigma_n^{\mathrm{ND}} \cap \Pi_n^{\mathrm{ND}} \\
\Sigma_{n+1}^{\mathrm{ND}} &=_{\mathrm{df}} \{P \subseteq U_{\mathcal{A}}^{\infty} \mid (\exists Q \in \Pi_n^{\mathrm{ND}}) \\
&\qquad \forall \vec{x}(\vec{x} \in P \Leftrightarrow (\exists \vec{y} \in U_{\mathcal{A}}^{\infty})((\vec{y}\,.\,\vec{x}) \in Q))\}
\end{aligned}
$$

$$\Rightarrow \quad \Sigma_{n+1}^{\mathrm{ND}} = \{P \subseteq U_{\mathcal{A}}^{\infty} \mid (\exists Q \in \Pi_n^{\mathrm{ND}})(P \in (\mathrm{SDEC}_{\mathbb{R}}^{\mathrm{ND}})^{Q})\}$$

Proposition (CCC 2015, G.)

$$
\begin{aligned}
\Sigma_{n+1}^{\mathrm{ND}} &= (\mathrm{SDEC}_{\mathcal{A}}^{\mathrm{ND}})^{(\mathbb{H}_{\mathcal{A}}^{\mathrm{ND}})^{(n)}} = \{P \subseteq U_{\mathcal{A}}^{\infty} \mid P \preceq_1 (\mathbb{H}_{\mathcal{A}}^{\mathrm{ND}})^{(n+1)}\} \\
\Pi_{n+1}^{\mathrm{ND}} &= \{P \subseteq U_{\mathcal{A}}^{\infty} \mid P \preceq_1 U_{\mathcal{A}}^{\infty} \setminus (\mathbb{H}_{\mathcal{A}}^{\mathrm{ND}})^{(n+1)}\}
\end{aligned}
$$

Corollary

Close relationship: nondeterministic BSS RAM's — Moschovakis' model.

Complete Problems for Structures $\mathcal{A}$
(Written in Blue)

1. Hierarchy:

$$
\begin{array}{ccccccccc}
 & & & & \vdots & & & & \\
\mathrm{FIN}_{\mathbb{N}} & \in \Sigma^0_2 & \nearrow\atop\nwarrow & & & & \nwarrow\atop\nearrow & \Pi^0_2 \ni & \mathrm{TOTAL}_{\mathbb{N}}, \mathrm{INCL}_{\mathbb{N}} \\
 & & & & \Delta^0_2 & & & & \\
\mathbb{H}_{\mathcal{A}} & \in \Sigma^0_1 & \nearrow\atop\nwarrow & & & & \nwarrow\atop\nearrow & \Pi^0_1 & \\
 & & & & \Delta^0_1 & & & &
\end{array}
$$

(MSCS 2016, G.)

2. Hierarchy

$$
\begin{array}{ccccccccc}
 & & & & \vdots & & & & \\
\mathrm{TOTFIN}^{\mathrm{ND}}_{\mathcal{A}} & \in \Sigma^{\mathrm{ND}}_3 & \nearrow\atop\nwarrow & & & & \nwarrow\atop\nearrow & \Pi^{\mathrm{ND}}_3 & \\
 & & & & \Delta^{\mathrm{ND}}_3 & & & & \\
\mathrm{FIN}^{\mathrm{ND}}_{\mathcal{A}} & \Sigma^{\mathrm{ND}}_2 & \nearrow\atop\nwarrow & & & & \nwarrow\atop\nearrow & \Pi^{\mathrm{ND}}_2 \ni & \mathrm{TOTAL}^{\mathrm{ND}}_{\mathcal{A}}, \mathrm{INCL}^{\mathrm{ND}}_{\mathcal{A}}, \mathrm{CONST}^{\mathrm{ND}}_{\mathcal{A}} \\
 & & & & \Delta^{\mathrm{ND}}_2 & & & & \\
\mathbb{H}^{\mathrm{ND}}_{\mathcal{A}} & \in \Sigma^{\mathrm{ND}}_1 & \nearrow\atop\nwarrow & & & & \nwarrow\atop\nearrow & \Pi^{\mathrm{ND}}_1 \ni & \mathrm{INJ}^{\mathrm{ND}}_{\mathcal{A}} \\
 & & & & \Delta^{\mathrm{ND}}_1 & & & &
\end{array}
$$

(CCC 2015, G.)

Thank you very much for your attention!

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