Recognizability strength of infinite time Turing machines

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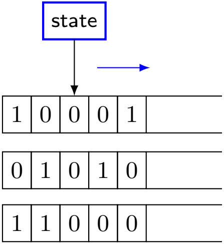
An infinite-time Turing machine (ITTM) is a Turing machine with three tapes – each has one cell for each natural number.

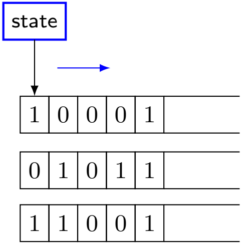
- Input tape
- Working tape
- Output tape

It behaves like a standard Turing machine at successor steps of computation.



At limit steps of computation

- The head goes back to the first cell
- The machine goes into a designated limit state
- The contents of each cell is set to the lim inf of the contents at previous stages of computation

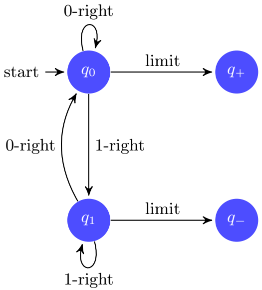


The snapshot at a fixed time consists of the tape contents, head position and state.

time	state	head	0	1	2	3	4	5	⋯
0	0	0	–	–	–	–	–	–	⋯
1	1	1	1	–	–	–	–	–	⋯
2	0	2	1	–	–	–	–	–	⋯
3	1	3	1	–	1	–	–	–	⋯
4	0	4	1	–	1	–	–	–	⋯
5	1	5	1	–	1	–	1	–	⋯
6	0	6	1	–	1	–	1	–	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
ω	0	0	1	–	1	–	1	–	⋯
$\omega+1$	1	1	0	–	1	–	1	–	⋯
$\omega+1$	0	2	0	–	1	–	1	–	⋯
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$\omega\cdot 2$	0	0	0	–	0	–	0	–	⋯
$\omega\cdot 2+1$	1	1	1	–	0	–	0	–	⋯
$\omega\cdot 2+2$	0	2	1	–	0	–	0	–	⋯

Example

Does the letter 0 appear infinitely often in the input word?



Example

Compute the halting problem for standard Turing machines by simulating all standard Turing machines.

Strength of *ITTMs*

Theorem (Hamkins-Lewis)

An ITTM can check whether a relation X on the integers - given as a set of codes for pairs (m, n) - is a well-order

Proof sketch.

- Check whether the relation is a linear order
- Search for the least element
- If this fails, the relation is not a well-order
- If this succeeds, delete the least element from the domain
- If the domain is empty, the relation is an ordinal

□

Decidable sets of reals

Definition

A set X of reals is decided by an *ITTM* P if P^x halts with state 1 for all $x \in X$ and P^x halts with state 0 for all $x \notin X$.

Theorem (Hamkins-Lewis)

All Π^1_1 sets of reals are ITTM-decidable.

Writable reals

Definition

A real x is *ITTM*-writable if there is an *ITTM* P such that on empty input, P halts with output x.

Theorem (Hamkins-Lewis)

All Π^1_1 reals are ITTM-writable.

Theorem

The Δ^1_1 reals are exactly those reals that are writable by an ITTM with bounded memory.

Writable ordinals

Definition

1. An ordinal is *ITTM*-writable if it is coded as a well-order by some *ITTM*-writable real.
2. Let λ denote the supremum of the writable ordinals.

Example

$\omega_1^{ck}, \omega_2^{ck}, \ldots$ are writable.

Eventually writable ordinals

λ is not writable, but it is *eventually writable*.

Definition

1. A real x is eventually writable if there is an *ITTM* whose output stabilizes at x.
2. An ordinal is eventually writable if it is coded as a well-order by some eventually writable real.
3. Let ζ denote the supremum of the eventually writable ordinals.

Lemma

λ is eventually writable.

Proof sketch.

- There is a universal *ITTM U* that simulates all *ITTMs*.
- In each step, calculate the sum of all ordinals that are coded by outputs of machines that have halted.
- This real will stabilize at a code for λ

□

Accidentally writable ordinals

ζ is not eventually writable, but it is *accidentally writable*.

Definition

1. A real x is accidentally writable if there is an *ITTM* that has x on its output tape at some time of the computation
2. An ordinal is accidentally writable if it is coded as a well-order by some accidentally writable real
3. Let Σ denote the supremum of the eventually writable ordinals

Lemma

ζ is accidentally writable.

Proof sketch.

- Consider a universal *ITTM* U that simulates all *ITTMs*
- In each step, calculate the sum of all ordinals that are coded by outputs of machines
- After codes for all eventually writable ordinals appeared, a code for some ordinal $\alpha \geqslant \zeta$ appears
- The accidentally writable ordinals are downwards closed

Clockable ordinals

Definition

An ordinal is clockable if it is the halting time of an *ITTM*-computation.

Lemma

Any contents that appears at time ω_1 appears at some countable limit time.

Proof.

- Choose a countable α_0 such that for every cell whose contents converges, it does so before α_0.
- Choose α_{n+1} such that the contents of all other cells changed at least once in $[\alpha_n, \alpha_{n+1})$.
- The contents at the time α is equal to the contents at the time ω_1.

□

Clockable ordinals

Theorem (Welch)

Any computation runs into a loop between ζ *and* Σ.

Theorem (Welch)

The supremum of the clockable ordinals is equal to λ.

Infinite time Turing machines with ordinal parameters

Definition

Assume that X is a class of ordinals.

- *An X-ITTM works like an ITTM...*
- *...with a special reserved state at running times in X*

We write α-ITTM for $X = \{\alpha\}$.

When X is the class of cardinals, these are the *cardinal-recognizing ITTMs* of Habic.

Theorem (Habic)

Cardinal-ITTMs can write the same reals as ITTMs with the strong halting problem $0^{\blacktriangledown} = \{(n, x) \mid \varphi_n(x)\downarrow\}$ for ITTMs as an oracle.

Proposition

An ω_1-ITTM can write a code for Σ.

Proof.

- U simulates all *ITTM*-programs simultaneously
- Add all ordinals written on the tape
- At time ω_1 we obtain a code x for ζ
- We claim that Σ is x-clockable
- To see this, we run U and count to ζ - then wait until the configuration repeats at Σ
- Hence $\lambda^x > \Sigma$ and Σ is x-writable
- Combine the two programs

□

Strength of *ITTMs* with ordinal parameters

Proposition

The following statements are equivalent for a real x.

1. *x is α-ITTM-writable for some ordinal α.*
2. *x is ITTM-writable from some accidentally writable real number.*
3. *x is an element of L_{λ^z}, where z is the L-least code for ζ.*

Proof.

$1 \Rightarrow 2$
At time α, the tape contains some accidentally writable real number – the remaining computation is an ordinary *ITTM*-computation.
$2 \Rightarrow 1$
If x is accidentally written at time α and P writes y from x, then y is α-writable.
$2 \Rightarrow 3$
Assume that x is writable from an accidentally writable real y. Since z codes ζ, we have $\lambda^z > \zeta$ and hence $\lambda^z > \Sigma$ – thus y is writable from z.
$3 \Rightarrow 2$
Since $z \in L_\Sigma$, it is accidentally writable. □

This generalizes to finitely many ordinals.

Proposition

The following statements are equivalent for a real x.

1. x *is ITTM-writable from* n *ordinals.*
2. x *is ITTM-writable from* x_{n-1}*, where* $x_0, \ldots, x_{n-1}$ *are reals with* x_j *is accidentally writable from* $\bigoplus_{i<j} x_i$ *for all* $j < n$*.*
3. x *is an element of* $L_{\lambda^{z_{n-1}}}$*, where* $z_0 = 0$ *and* z_{i+1} *is the L-least code for* ζ^{z_i} *for all* $i < n - 1$*.*

Question

Which reals are writable by cardinal-detecting ITTMs?

Definition

1. *A real x is ITTM-recognizable relative to a real y and an ordinal α if for some α-ITTM-program P*
   - $P^{x \oplus y} = 1$
   - $P^{x \oplus y} = 0$ *for all* $x' \neq x$
2. *The ITTM-recognizable closure $\mathcal{R}$ is the closure under relativized recognizability.*

Clearly every writable real is recognizable.

Question

Which reals are ITTM-recognizable?

Lemma

No $x \in L_\Sigma \setminus L_\lambda$ is ITTM-recognizable.

Proof.

Assume that $x \in L_\Sigma$ is recognizable by an *ITTM*-program P.

- Consider an *ITTM*-program U that writes every accidentally writable real at some time
- We run U and for each tape contents, run P to check whether it is equal to x
- In this case stop and output x – hence x is writable

□

Definition

Let σ be least with $L_\sigma \prec_{\Sigma_1} L$.

Proposition

There are unboundedly many ordinals α below σ such that the L-least code for L_α is ITTM-recognizable.

Proof.

Assume that φ is a Σ_1-statement that is first true in L_α and x is its L-least code – this code is an element of $L_{\alpha+1}$.

- Check if the input y codes an ordinal γ
- Construct a code for $L_{\gamma+1}$
- Check if φ is first true in L_γ and if y is its L-least code in $L_{\gamma+1}$.

□

Theorem

Every ω_1-recognizable real x is an element of L_σ.

Proof.

- Assume that P recognizes x and P^x halts with the final state s
- Assume that y is a subset of ω
- The configurations at ζ^y, Σ^y and ω_1 are identical
- Let $c_{y,\alpha}$ denote the $L[y]$-least code for any α that is countable in $L[y]$
- If P^y with the parameter Σ^y halts, then it does strictly before $\lambda^{y \oplus c_{y,\Sigma^y}}$
- Let $\varphi(y)$ be the statement that P^y with the parameter Σ^y halts in $L_{\lambda^{y \oplus c_{y,\Sigma^y}}}[y]$ with final state s
- This is a Σ_1-statement and hence there is such a y in L_σ
- We have $x = y$

□

Definition

1. *A real x is ITTM-recognizable relative to a real y and an ordinal α if for some α-ITTM-program P*
   - $P^{x \oplus y} = 1$
   - $P^{x \oplus y} = 0$ *for all* $x' \neq x$
2. *The ITTM-recognizable closure $\mathcal{R}$ is the closure under relativized recognizability.*

Clearly every writable real is recognizable.

Question

Which reals are ITTM-recognizable?

Proposition

There are unboundedly many countable ordinals α such that the L-least code for L_α is α-recognizable.

Theorem

Every ITTM-recognizable real from finitely many ordinal parameters is an element of L.

Steps of the proof.

- Assume that x is recognizable from n ordinal parameters
- Show that it is recognizable from parameters below $\omega_1(n+1)$
- Show that this implies $x \in L$ by a forcing argument

□

Hence the *ITTM*-recognizable closure with ordinal parameters is equal to the set of reals in L.

Theorem

For every n, there is a real x that is ITTM-recognizable from $n+1$ ordinal parameters, but not n parameters.

Open questions

Question

Assuming that X is a set of cardinals, is every X-recognizable real in L_σ?

Question

Is every real that is ITTM-recognizable from some ordinal already ITTM-recognizable from some countable ordinal?

Question

What can we cay about semi-recognizable and co-semi-recognizable reals?