On the Computability and Reducibility of Approximable Real Numbers

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Computable Real Numbers (EC)

- A real x is computable if there is a computable sequence (x_s) of rationals which converges to x effectively, i.e., $|x_s - x_{s+1}| \leq 2^{-s}$. (**EC** – class of **e**ffectively **c**omputable reals)

- x is computable iff
  - its Dedekind cut $D_x := \{r \in \mathbf{Q} : r < x\}$ is computable
  - its binary expansion A is computable, where $x = x_A := \sum_{i \in A} 2^{-(i+1)}$ $(x = 0.A)$
  - continued fractions
  - nested intervals
  - $\ldots\ldots$

- **EC** is a field.

- **EC** is closed under computable real functions (computable operations).

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Computably Enumerable Real Numbers (CE)

- x is c.e. (co-c.e.) if there is a computable increasing (decreasing) sequence (x_s) of rationals which converges to x.

- (Calude, Hertling, Khoussainov and Wang 2001) x_A is c.e. iff A is strongly ω-c.e., i.e. A has a computable approximation (A_s) of finite sets such that

$$(\forall n)(\forall s)(n \in A_s \setminus A_{s+1} \Longrightarrow (\exists m < n)(m \in A_{s+1} \setminus A_s)$$

- If x_A is c.e. and additionally that A is d-c.e. or h-c.e., then it is stably c.e. (Soare 1969), or h-stably c.e. (Weihrauch and Z. 1997).
  There is an Ershov's hierarchy of h-stably c.e. reals up to the level of 2^n-stably c.e.

- If x_A is c.e. A is c.e., then x_A is strongly c.e. (Downey and Hu 2003), and x is k-strongly c.e. if it is the sum of k strongly c.e. reals. x is regular if it is k-strongly c.e. for some k. The hierarchy theorem holds.

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Semi-Computable Real Numbers (SC)

- x is semi-computable if it is either c.e. or co-c.e.

- x is semi-computable if there is a computable sequence (x_s) of rationals which converges to x (1-)monotonically: $|x - x_t| \leq |x - x_s|$ for all $t \geq s$.

  (h-monotonically computable for $|x - x_t| \leq h(s)|x - x_s|$ for all $t \geq s$)

- (Ambos-Spies, Weihrauch, Z. 2000) If $x_{A \oplus \overline{B}}$ is semi-computable and A, B are c.e. sets , then A and B must be Turing comparable.

- There are (strongly) c.e. reals x and y such that $x - y$ is not semi-computable.

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D.C.E. Real Numbers (DCE)

- x is d.c.e. if $x = y - z$ for two c.e. reals y and z.

- (Ambos-Spies, Weihrauch, Z. 2000) x is d.c.e. iff there is a computable sequence (x_s) of rational numbers which converges to x weakly effectively, i.e. $\sum |x_s - x_{s+1}|$ is finite.

- SC $\subsetneq$ DCE and DCE is a field.

- x is d.c.e. iff there is a computable sequence (x_s) of rationals which converges to x c.e. bounded., i.e., $|x - x_s| \leq \sum_{i \geq s} \delta_i$, where (δ_i) is a computable positive rationals with a finite sum $\sum \delta_i$.

- (Rettinger, Z. 2005) If x is c.e. and random, then it is either c.e. or co-c.e.

- (Ambos-Spies, Weihrauch Z. 2000) If x_{2A} is d.c.e., then A must be a 2^{3n}-c.e. set.

- (Rettinger and Z. 2001) There is a c.e. real x and a computable total real function f such that $f(x)$ is not d.c.e.

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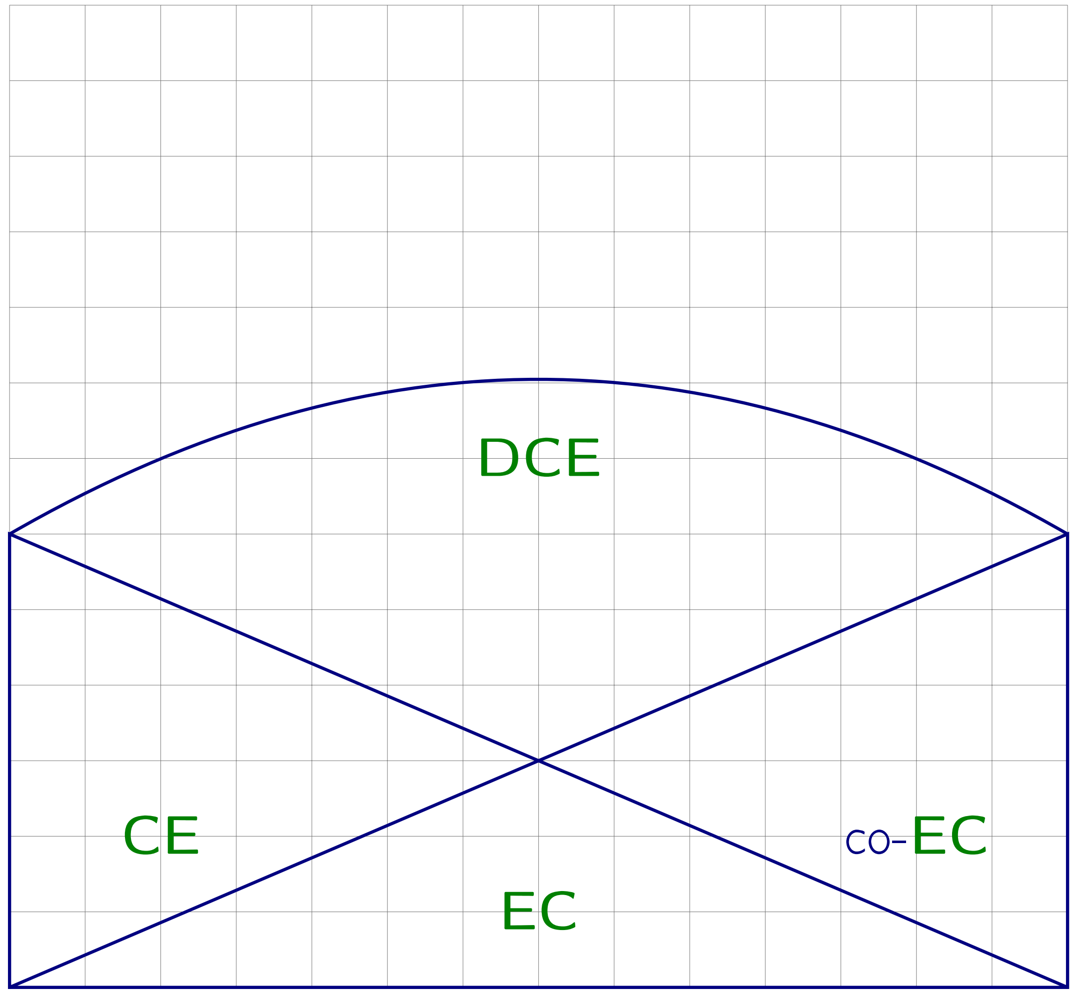
DCE
CE
co-EC
EC

D.B.C. Real Numbers (DBC)

- x is dbc (divergence bounded computable) iff there is a dce real y and a total computable real function f such that $x = f(y)$. **DCE** $\subsetneq$ **DBC**.

- x is h-e.c. if there is a computable sequence (x_s) of rationals which converges to x h-effectively, i.e., there are atmost $h(n)$ non-overlapped index-pairs (i, j) with $i, j \geq n$ and $|x_i - x_j| \geq 2^{-n}$. (h-**EC**)

- x is bec (bounded effectively computable) if it is k-ec for some constant k. (**BEC**)

- (Z. 2008)
  - k-**EC** $\subsetneq (k+1)$-**EC** $\subsetneq$ **BEC** $\subsetneq$ **DBC**
  - $(\exists^\infty n)(f(n) < g(n)) \Longrightarrow g$-**EC** $\not\subseteq f$-**EC**.
  - If C is a class of functions which contains all constant functions and is closed under composition, then C-**EC** is a field.

- (Rettinger and Z. 2001) x is dbc iff there is a computable function h such that x is h-e.c.

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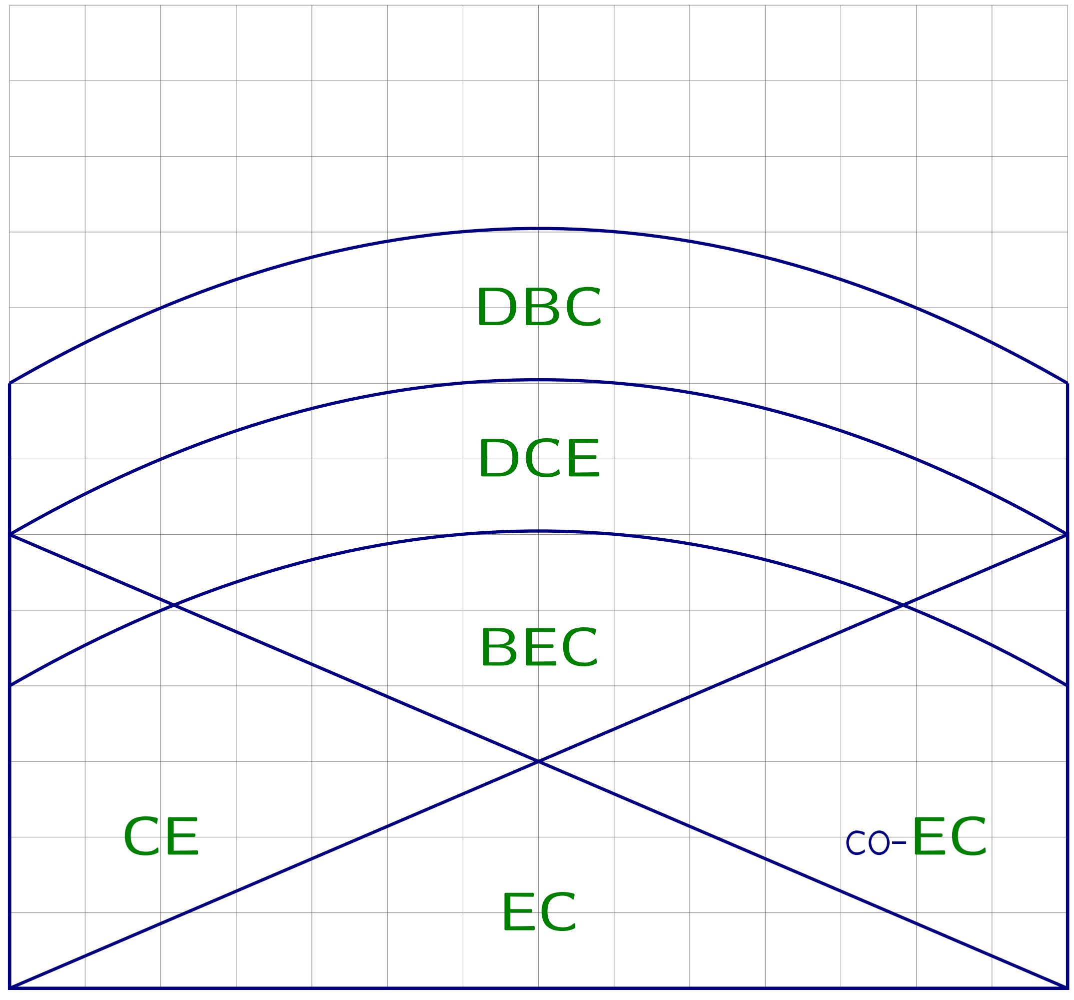
DBC
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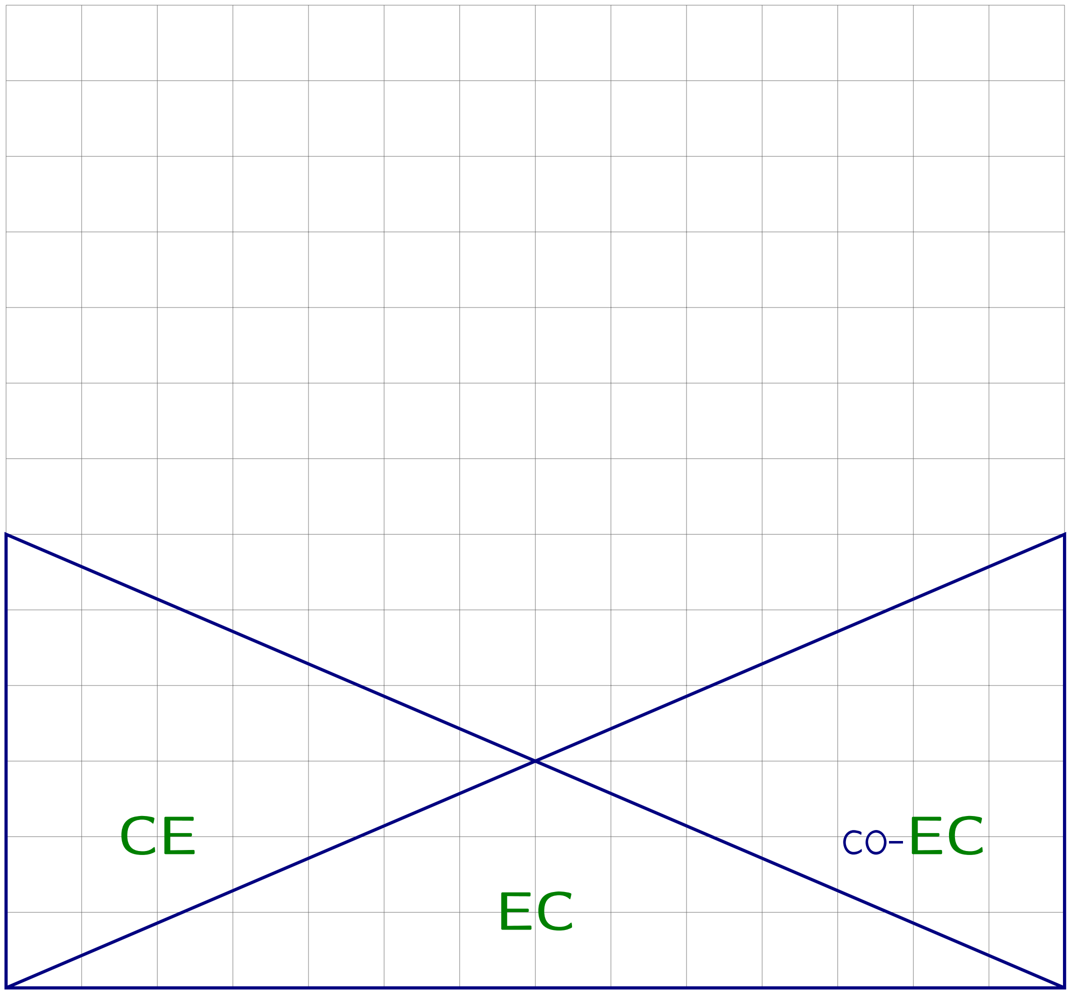
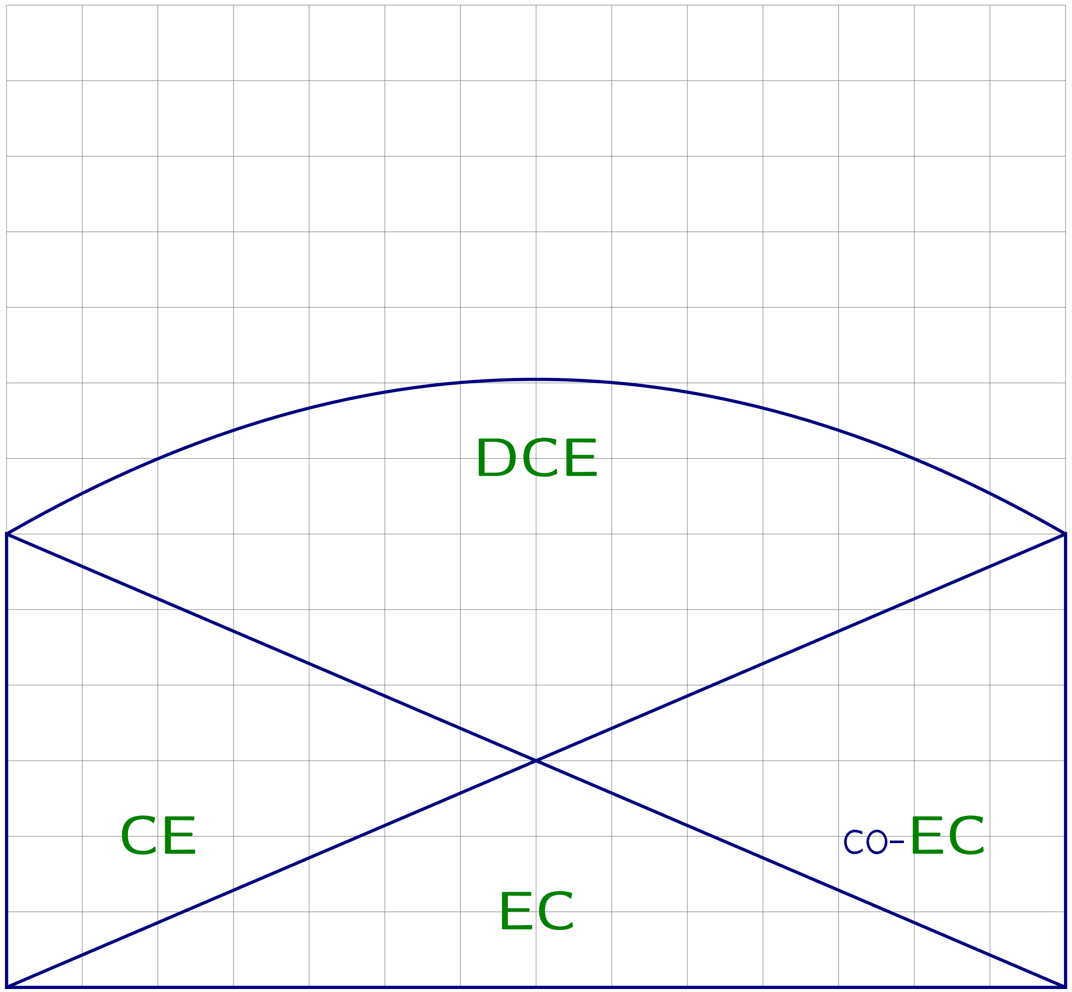
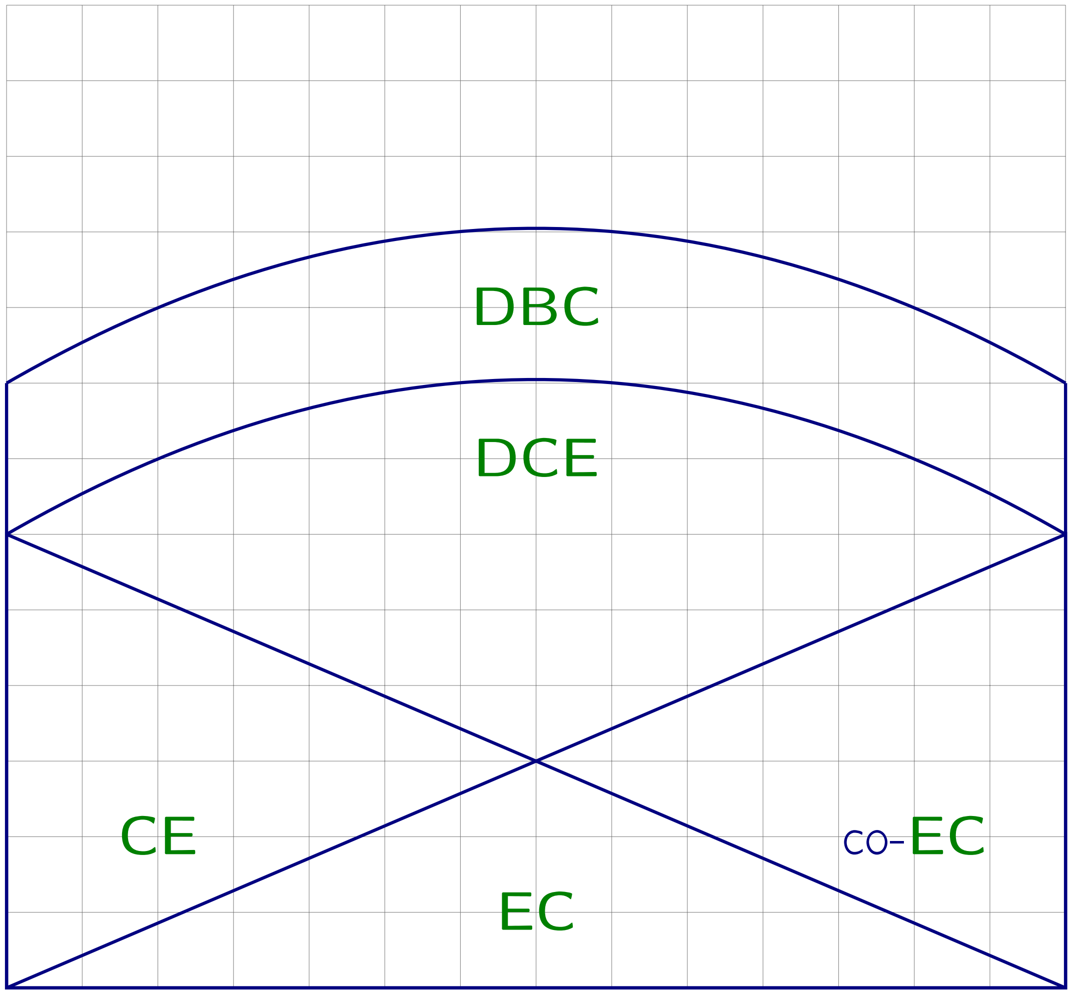
Computably Approximable Real Numbers (CA)

- x is c.a. (computably approximable) if it is the limit of a computable sequence of rationals.
- **CA** is a field, closed under computable function and **DBC** $\subsetneq$ **CA**


CE
EC
co-EC


DCE
CE
co-EC
EC


DBC
DCE
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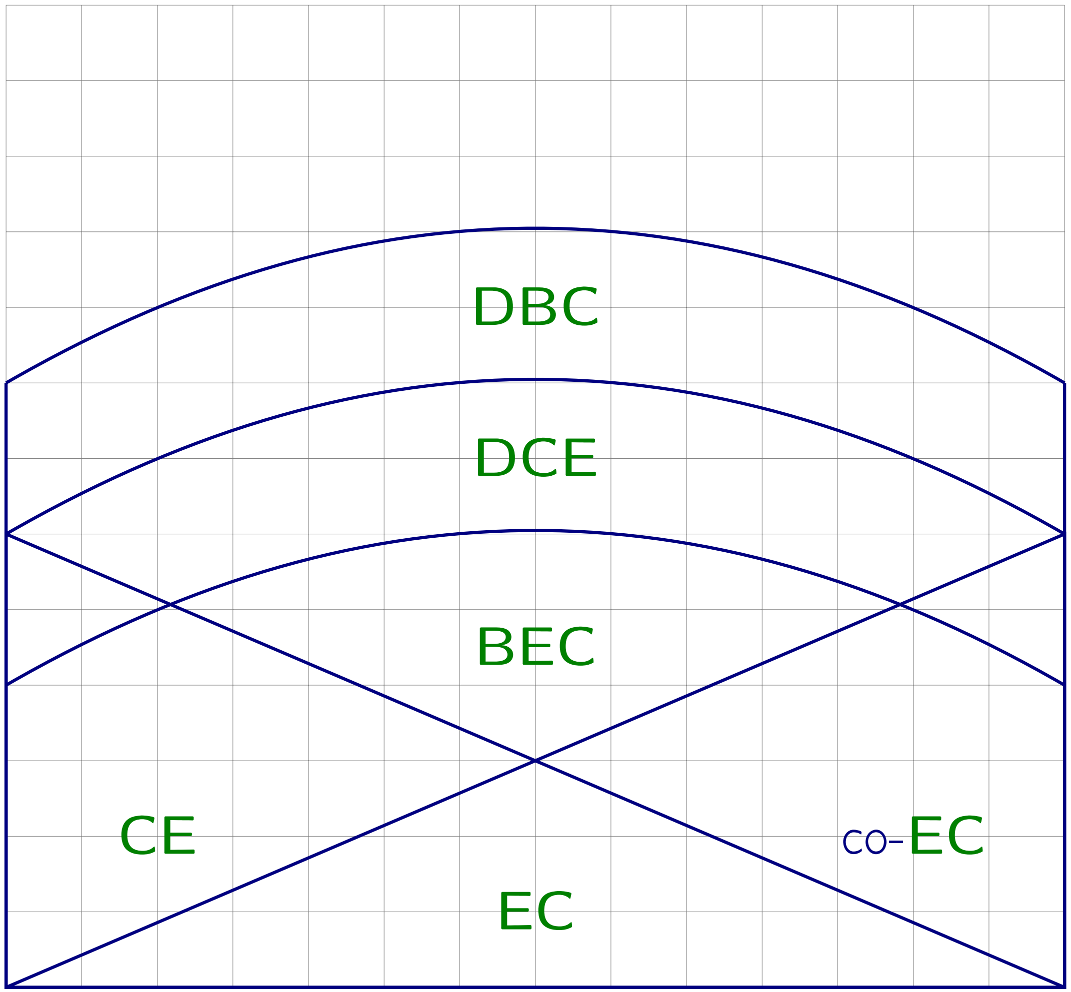
DBC
DCE
BEC
CE
co-EC
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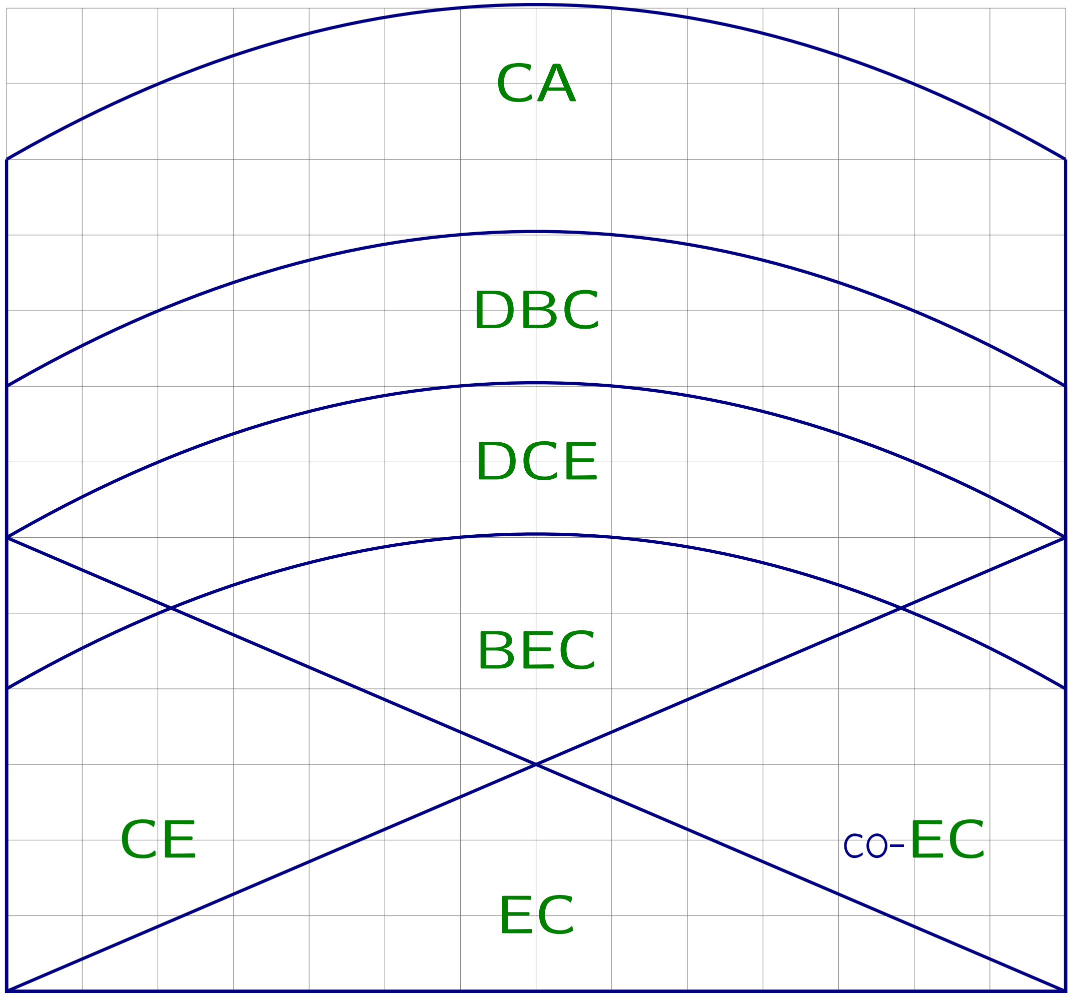
CA
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Turing Reucibility Of Real Numbers

- $A \leq_T B$ if there is a Turing machine Φ such that $A = \Phi^B$
- $x \leq_T y$ if there are sets A and B such that $A \leq_T B$ & $x = x_A$ & $y = x_B$
- (Ko 1984) $x \leq_{RF} y$ (and $x \leq_{IRF} y$) if there exist computable (increasing) real function f such that $f(y) = x$. Then,
  - $x \leq_{RF} y \Rightarrow x \leq_T y$, but $x \leq_T y \not\Rightarrow x \leq_{RF} y$
    (Computable modulus function of computable real function matters)
  - $x \leq_{RF} y \Leftrightarrow x \leq^R_{wtt}$
  - $x \leq_{IRF} y \Leftrightarrow x \leq^R_{tt}$

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Turing Degrees Of Real Numbers

- $\mathbf{D}(C)$ — set of T-degrees of some set $A \in C$. (c.e. degrees, ω-c.e. degrees, etc.)

- $\mathbf{D}_R(C)$ — set of degrees of reals $x \in C$. (degree of c.e. reals, degree of dbc reals, etc)

- There is a degree if c.e. real which is not a c.e. degree. $(\mathbf{D}(\mathbf{CE}) \subsetneq \mathbf{D}_R(\mathbf{CE}))$

- (Z. 2003) There are two strongly c.e. reals x and y such that $\deg_T(x-y)$ is not ω-c.e. $(\mathbf{D}(\mathbf{DCE}) \subsetneq \mathbf{D}_R(\mathbf{DCE}))$

- (Downey, Hu and Z. 2004)
  - Every ω-c.e. degree contains a d.c.e. real, (hence $\mathbf{D}(\omega\text{-}\mathbf{CE}) \subsetneq \mathbf{D}_R(\mathbf{DCE})$)
  - There is a Δ^0_2-degree which is not a degree of d.c.e real. $(\mathbf{D}_R(\mathbf{DCE}) \subsetneq \mathbf{D}(\mathbf{CA}))$

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Solovay Reducibility

- (Original Solovay reduction) $x \leq_S^0 y$ if there are increasing computable sequences (x_s) and (y_s) of rationals and a constant c such that $x - x_s \leq c(y - y_s)$ for all s. (On **CE** only!)
- (Solovay property) $x \leq_S^0 y \Longrightarrow K(x \upharpoonright n) \leq K(y \upharpoonright n) + c$ for some c and all n, where $K(\sigma) = \min\{|\tau| : U(\tau) = \sigma\}$ for a universal prefix-free machine U.
- x is random if $K(x \upharpoonright n) \geq n - c$ for a c and all n.
- $\Omega_U := \sum\{2^{-\sigma} : U(\sigma)\downarrow\}$ where U is a prefix-free universal Turing machine. (Ω-numbers)
- x is Ω-like if x is complete for **CE** under $\leq_S^0$.
- Ω-numbers $\approx$ Ω-like $\approx$ c.e. random
  - (Solovay 1975) Ω-number $\Longrightarrow$ Ω-like $\Longrightarrow$ c.e. random
  - (Calude et al 2001) Ω-like $\Longrightarrow$ Ω-number
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Solovay Reducibility on DCE

- $x \leq_S^1 y$ if there are computable sequences (x_s) and (y_s) of rationals which converges to x and y, respectively, such that $|x - x_s| \leq c(|y - y_s| + 2^{-s})$ some c and for all s
- $\leq_S^0 \equiv \leq_S^1$ on **CE** and $\leq_S^1$ has the Solovay property
- $y \in$ **DCE** & $x \leq_S^1 y \Longrightarrow x \in$ **DCE**
- (Retinger and Z. 2005)
  - If $f : \mathbf{R}^n \to \mathbf{R}$ is computable and locally Lipschitz, and let d be c.a., then $S(\leq d) := \{x : x \leq_S^1 d\}$ is closed under f.
  - For any c.a. real d, $S(\leq d)$ is a field.
  - If d is c.e. random, then $S(\leq d) =$ **DCE**

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Convergence-Bounded Reducibility

- $x \leq_{cd} y$ if there is computable monotone real function h with $h(0) = 0$, and two computable sequences (x_s) and (y_s) of rationals which converge to x and y, respectively, such that $|x - x_s| \leq h(y - y_s|)$ for all s.

- (Rettinger and Z. 201?)
  - $x \leq_{cd} y$ iff there is a computable function $k : \mathbf{N} \to \mathbf{N}$ which is non-decreasing and unbounded, and there are two computable sequences (x_s) and (y_s) of rationals which converges to x and y, respectively, such that

$$(\forall n)(|y - y_s| \leq 2^{-n} \Longrightarrow |x - x_s| \leq 2^{-k(n)}).$$

  - $x \leq_S y \Longrightarrow x \leq_{cd} y$
  - $y \in \mathsf{DBC}$ & $x \leq_{cd} y \Longrightarrow x \in \mathsf{DBC}$
  - $x \in \mathsf{DBC} \iff x \leq_{cs} \Omega$

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Thank You for Attention