COMPUTABILITY AND LOGIC

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Hannes Diener
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Christine Gaßner
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Philipp Schlicht
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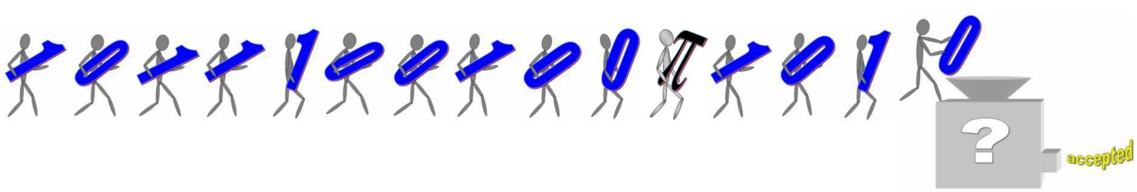
Isabel Schwende
&
Dieter Spreen



Meeting "Computability and logic"

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Continuity in Constructive Mathematics

Hannes Diener

Universität Siegen

FB Mathematik, Mathematische Logik und Theoretische Informatik

As (hopefully) every student of mathematics learns in their first semester there are various notions of continuity, such as sequential continuity, normal (pointwise) continuity, uniform continuity, and some more. Furthermore the student learns that all of these notions coincide for functions $[0, 1] \to \mathbb{R}$. However, the proofs that these notions are equivalent make use of non-constructive principles, making the extraction of algorithms from proofs often impossible. Working constructively one therefore constantly faces a choice between various competing notions. In this talk we will give an overview of how various schools of constructive mathematics have tried to deal with this problem and discuss the baggage that comes with the various attempts to simplify matters. We will also present some recent work showing that, surprisingly, in many situation assuming continuity is actually unnecessary from a constructive point of view.

On the Axiom of Choice and Infinite Graphs

Christine Gaßner

Ernst-Moritz-Arndt-Universität Greifswald
Institut für Mathematik und Informatik

The De Bruijn-Erdős theorem (1951) states that an infinite graph G has the chromatic number at most k if every finite subgraph of G can be colored by k colors. In the Zermelo-Fraenkel set theory with the axiom of choice (ZFC) many proofs of this theorem are discussed and all proofs of this theorem use a statement which can be derived from the axiom of choice. A consequence of the De Bruijn-Erdős theorem is that any infinite graph can be colored by 4 colors. This sentence and many other statements such as the axiom of choice can be formalized in the second-order logic and, in connection with the Henkin interpretations (HPL), theorems such as Bernstein's equivalence theorem and the fixed-point theorem of Bourbaki are provable, the axiom of choice follows from the well-ordering theorem, Zorn's lemma follows from the axiom of choice, etc. On the other hand, the well-ordering theorem for unary predicates is independent of axiom of choice for binary predicates and of the trichotomy law for unary predicates. Thus the question arises whether the statement that any infinite graph can be colored by a finite number of colors can be derived from the axiom of choice or the trichotomy law in the second order logic. We answer the question in the negative by providing some counterexamples.

1. R. Diestel: Graph Theory, Springer (2006).
2. C. Gaßner: The Axiom of Choice in Second-Order Predicate Logic, Mathematical Logic Quarterly 40 (1994), 533 – 546.

Analytically Computable Functions

Christine Gaßner

Ernst-Moritz-Arndt-Universität Greifswald
Institut für Mathematik und Informatik

The analytic machines introduced by Günter Hotz are register machines that extend BSS machines by infinite converging computations. This model can be used to characterize the computability of analytic functions (as e.g. in [1]). The structure underlying the model can be the field $\mathbb{R}$ or $\mathbb{C}$ of the real and complex numbers, respectively. In any registers a real or a complex number can be stored and, moreover, a machine can perform the permitted operations and comparisons on real or complex numbers in a fixed time unit. Infinite computations are valid if an output is written infinitely often. Let $\mathcal{M}^{(n)}(x)$ be the n-th output of a machine $\mathcal{M}$ on input x. If the sequence of outputs of an infinite computation is convergent, then $\lim_{n\to\infty} \mathcal{M}^{(n)}(x)$ exists and the computation is *analytic*. $\mathcal{M}^{(n)}(x)$ is the n-th approximation of the computation of $\mathcal{M}$. Here, we want to discuss several questions resulting from the definition of decidability by analytic machines.

1. T. Gärtner: Representation Theorems for Analytic Machines. In: A. Beckmann, C. Gaßner, B. Löwe (eds.): Logical Approaches to Barriers in Computing and Complexity, International Workshop, Greifswald, Germany, February 2010 (Preprint 6/2010), 117 – 120.

Ordinal Automatic and Tree Automatic Structures

Philipp Schlicht

Rheinische Friedrich-Wilhelms-Universität Bonn
Mathematisches Institut

A structure in a finite signature is automatic if its domain consists of finite words and the domain, relations, and functions of the structure are recognized by finite automata. Automatic presentations can be generalized to ordinal tapes and to tree automata, and there is a close connection between these. I will describe how to prove non-automaticity in both settings. For example, in joint work with Jain, Khoussainov, and Stephan we prove that the ranks of linear orders with an ω^n-automatic presentation or a tree-automatic presentation are finite and that there is no scattered tree-automatic linear order on the set of all finite labelled trees.

On the Wang Machine and the Minimality of Instructions of Turing Machines

Isabel Schwende

Ernst-Moritz-Arndt-Universität Greifswald
Institut für Mathematik unfd Informatik

The paper of Alan Turing "On Computable Numbers, with an Application to the Entscheidungs problem" (1936) was the dawn of the era of theoretical machine models. In 1957 Hao Wang published his research on a variant of the turing machine. The main achievement was the implementation of only four basic instructions without loosing any power in terms of computability. The gain of a small set of instructions is mainly the lower effort for complex proofs but the downside is obviously an increase of the length and intricacy of actual programs. This gain is useful for proving machines because simple proofs are enormously valuable. Furthermore is there an impact evident in area of register machines because Wang was one of the first researchers who was concerned by the unravelling of practical and theoretical computer science.

An Isomorphism Theorem for Partial Numberings

Dieter Spreen

Universität Siegen
FB Mathematik, Mathematische Logik und Theoretische Informatik

As has been shown by the author, standard numberings of the computable real numbers and similar effectively given topological spaces are only partially

defined, by necessity. Thus, not every natural number is a name of some computable object. It was demonstrated that any two such numberings are m-equivalent. Spaces like the partial computable functions, on the other hand, are known to have totally defined standard numberings such that any two of them are even recursively isomorphic.

In this talk it is studied whether such a result is also true for standard numberings of the computable reals and similar spaces. The investigation is carried out in the general setting of effective topological spaces introduced by the author. For total numberings it is well known that m-equivalent numberings are recursively isomorphic if they are precomplete. The proof proceeds in two steps: First it is shown that m-equivalent precomplete numberings are already 1-equivalent and then a generalization of Myhills theorem is applied. If one extends the usual reducibility relation between numberings to partial numberings in a straightforward way, the reduction function is allowed to map non-names with respect to one numbering onto names with respect to the other. A recursive isomorphism, however, can only map non-names onto non-names. If one allows only reduction functions operating in the same way – we speak of strong reducibility in this case –, the usual construction for Myhills theorem goes through and one obtains a generalization of this theorem to partial numberings.

In the second part of the talk the notion of admissible numbering of an effective space is strengthened in a similar way as was the reducibility notion. For the strongly admissible numberings thus obtained one has that any two of them are even strongly m-equivalent. A necessary and sufficient condition is presented for when such numberings are precompete.

As is finally shown, effective spaces have strongly admissible precomplete numberings. By the above results any two of them are recursively isomorphic.