Classification of Non-Positive Universal Products and their GNS-Modules

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Abstract

It is known that there are exactly five positive universal products. We classify non-positive universal products and find a two-parameter deformation of the Boolean product, which we call (r,s)-products. Furthermore, we introduce a GNS-construction for non-positive linear functionals on algebras and study the GNS-modules of (r,s)-product functionals.

1 Introduction

In classical probability theory, two random variables X and Y defined on the same probability space are *independent* if and only if

$$\mathbb{E}\big(f(X)g(Y)\big) = \mathbb{E}\big(f(X)\big)\mathbb{E}\big(g(Y)\big)$$

for all bounded measurable functions f and g. If X and Y are real-valued and bounded themselves, it is even enough that their joint moments are equal to the product of their respective moments, that is

$$\mathbb{E}(X^m Y^n) = \mathbb{E}(X^m)\mathbb{E}(Y^n) \tag{1}$$

for all $m, n \in \mathbb{N}$. This can be called a *universal calculation rule for mixed moments*, where "universal" roughly means that it does not depend explicitly on the random variables X and Y, but only on their respective moments. In quantum probability, random variables do not generally commute, so (1) is not sufficient to determine the joint distribution. As opposed to the classical situation, there are several different such universal calculation rules. Surprisingly, associativity limits the number of possibilities to a finite number. So a complete classification was possible and executed in three main steps:

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- Assuming the product is associative and commutative, Speicher showed in [18] that there are only three possibilities, namely the tensor product, the free product, and the Boolean product, thus proving a conjecture of Schürmann.
- Ben Ghorbal and Schürmann showed in [3] that Speicher's setting of universal calculation rules is equivalent to the categorial axioms we also use to define universal products (see Definition 3.1 and Theorem 3.2). They also classified all products satisfying their axioms in the categories of commutative unital algebras, commutative algebras, unital algebras, and general algebras.
- Dropping the commutativity condition, Muraki showed in [16] that there are exactly five universal products (see Theorem 3.3).

There are also weaker notions of non-commutative independence which do not directly fit into the above framework. Recently Muraki [15] has introduced a certain family of non-associative universal products to find an independence which is related to the q-Brownian motion of Bozeijko and Speicher [6]. Other examples are the matricial freeness of Lenczewski [13], c-freeness of Bozejko, Leinert and Speicher [5], bm-independence of Wysoczanski [23], differential independence of Hasebe [10] and there are even more. Since there is no general axiomatic framework, a classification including all of these concepts is out of reach.

Another important topic is independences for graded (or braided) algebras, see for example [7], [17] and [9], in particular Fermi-independence, which is important in physics. These are almost covered by the approach with universal products: There are so-called Bosonization theorems which provide us with a reduction of these independences to the tensor independence, see [8] and [14].

We use the term *algebraic quantum probability space*, meaning nothing but an algebra with a linear functional, which we call *expectation.* This is the minimum of structure we need to talk about moments and distributions. From the categorial viewpoint in [3] it does not even seem necessary to enforce a condition like (1). In fact, it can be shown that this condition is equivalent to a certain positivity condition, which makes sense, when working in the category of $*$-algebras with positive functionals, but not in the category of algebraic quantum probability spaces. Without such an axiom Ben Ghorbal and Schürmann find one parameter families associated with the tensor product, the free product, and the Boolean product, and show that these are all non-trivial commutative universal products in the category of algebras.

In this paper we drop the normalization condition in the non-commutative case. We find a new class of products, the (r,s)-products, which form a two-parameter deformation of the Boolean product. In Section 3 we show that the (r,s)-products are the only new products that satisfy our weakened set of axioms. For $r = s$ we can use the same argument that Ben Ghorbal and Schürmann use in [3] to reduce the non-normalized case to the normalized one. We repeat their argument in Observation 2 and apply it also to the non-commutative products.

The classification in the case $r \neq s$ is the main part of this paper. The only gap left in our classification is the case $r = s = 0$, which is surprisingly difficult. Up to now, this is an open problem. Actually, Ben Ghorbal and Schürmann state in [3] that there is exactly one commutative universal product with $r = s = 0$, but the proof contains a mistake, so we do not even know the answer in the commutative case. In Section 4 we present a construction similar to those in [7] to reduce (r,s)-independence to tensor-independence. Finally, we study the GNS-modules of the resulting product functionals in Section 5.
Some results from probability theory can also be formulated and proved for the (r,s)-products, as for the other universal products. In particular there are central limit theorems, and although these products do not preserve positivity, in some cases the moments of a classical probability distribution appear. This will be discussed in [11]. It is even possible to construct (r,s)-Lévy-processes as operator processes on Fock space, which is subject to ongoing research.

2 Preliminaries and Notations

Let I be an arbitrary index set. We put

$$\mathbb{A}_I := \{\varepsilon = (\varepsilon_1, \dots, \varepsilon_m) \mid m \in \mathbb{N}, \varepsilon_k \in I, \varepsilon_k \neq \varepsilon_{k+1}, k = 1, \dots, m-1\}$$

and define the *length of* ε as $|\varepsilon| := m$ if $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m)$. We simply denote $\mathbb{A}_{\{1,\dots,k\}}$ by $\mathbb{A}_k$. For $\varepsilon \in \mathbb{A}_I$, $|\varepsilon| = m$ and vector spaces $\mathcal{V}_i, i \in I$, we set

$$\mathcal{V}_\varepsilon := \mathcal{V}_{\varepsilon_1} \otimes \cdots \otimes \mathcal{V}_{\varepsilon_m}.$$

The *free product* of algebras $\mathcal{A}_i, i \in I$, is defined as the vector space

$$\bigsqcup_{i \in I} \mathcal{A}_i := \bigoplus_{\varepsilon \in \mathbb{A}_I} \mathcal{A}_\varepsilon$$

with the multiplication given by

$$(a_1 \otimes \cdots \otimes a_m)(b_1 \otimes \cdots \otimes b_n) := \begin{cases} a_1 \otimes \cdots \otimes a_m \otimes b_1 \otimes \cdots \otimes b_n, & \epsilon_m \neq \delta_1 \\ a_1 \otimes \cdots \otimes a_m b_1 \otimes \cdots \otimes b_n, & \epsilon_m = \delta_1 \end{cases}$$

for all $a_1 \otimes \cdots \otimes a_m \in \mathcal{A}_\varepsilon, b_1 \otimes \cdots \otimes b_n \in \mathcal{A}_\delta$, where $\varepsilon, \delta \in \mathbb{A}_I$, $|\varepsilon| = m, |\delta| = n$. By slight abuse of notation, expressions of the form $a_1 \cdots a_n \in \mathcal{A}_\varepsilon$ are always supposed to signify $|\varepsilon| = n$ and $a_i \in \mathcal{A}_{\varepsilon_i}$. Similarly $a_1 \cdots a_n \in \bigsqcup_{i \in I} \mathcal{A}_i$ shall mean there is an $\varepsilon \in \mathbb{A}_I$ such that $a_1 \cdots a_n \in \mathcal{A}_\varepsilon$, that is $a_1, \dots, a_n$ are always assumed to belong to alternating algebras. The free product of two algebras is denoted by $\mathcal{A}_1 \sqcup \mathcal{A}_2$ and has the following universal property: For two algebra homomorphisms $j_i : \mathcal{A}_i \to \mathcal{A}$, $i \in \{1, 2\}$, one gets a unique algebra homomorphism $j_1 \sqcup j_2 : \mathcal{A}_1 \sqcup \mathcal{A}_2 \to \mathcal{A}$ such that

$$j_1 \sqcup j_2 \, (a) = j_i(a)$$

for all $a \in \mathcal{A}_i$. In particular, this implies that for two algebra homomorphisms $j_i : \mathcal{A}_i \to \mathcal{B}_i$ we have a unique algebra homomorphism $j_1 \amalg j_2 : \mathcal{A}_1 \sqcup \mathcal{A}_2 \to \hat{\mathcal{A}}_1 \sqcup \hat{\mathcal{A}}_2$ fulfilling

$$j_1 \amalg j_2\,(a) = j_i(a)$$

for all $a \in \mathcal{A}_i$. When the algebras $\mathcal{A}_1, \mathcal{A}_2$ are unital, we define the *free product with identification of units* $\mathcal{A}_1 \sqcup_{\mathbb{1}} \mathcal{A}_2$ as the quotient of $\mathcal{A}_1 \sqcup \mathcal{A}_2$ by the ideal generated by $\mathbb{1}_{\mathcal{A}_1} - \mathbb{1}_{\mathcal{A}_2}$. This is indeed a unital algebra. The *unitization* of an algebra $\mathcal{A}$ is defined as $\widetilde{\mathcal{A}} := \mathbb{C}\widetilde{\mathbb{1}} \oplus \mathcal{A}$ with the unique multiplication which extends the multiplication of $\mathcal{A}$ such that $\widetilde{\mathbb{1}}$ is the unit of $\widetilde{\mathcal{A}}$. For an algebra homomorphism $f : \mathcal{A} \to \mathcal{B}$ define $\widetilde{f} : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{B}}$ as the unique unital algebra homomorphism whose restriction to $\mathcal{A}$ equals f. One canonically has

$$\widetilde{\mathcal{A}_1 \sqcup \mathcal{A}_2} \cong \widetilde{\mathcal{A}_1} \sqcup_{\mathbb{1}} \widetilde{\mathcal{A}_2}.$$

Let S be a set. The set of all tuples of arbitrary length over S is denoted by

$$S^* := \{\varnothing\} \cup \{(i_1, \ldots, i_n) \mid n \in \mathbb{N}, i_1, \ldots, i_n \in S\}.$$

We identify $S^n \times S^m$ with S^{n+m}, such that

$$(R, R') = \begin{cases} R & \text{for } R' = \varnothing, \\ R' & \text{for } R = \varnothing, \\ (i_1, \ldots, i_n, j_1, \ldots, j_m) & \text{for } R = (i_1, \ldots, i_n) \text{ and } R' = (j_1, \ldots, j_m). \end{cases}.$$

A *tuple partition* of S is a set of tuples $\Pi = \{V_1, \ldots, V_\ell\}, V_i \in S^* \setminus \{\varnothing\}$ such that every element of S belongs to one and only one of the tuples V_i. The set of all tuple partitions of S is denoted by $\mathrm{TP}(S)$. A tuple partition of $\{1, \ldots, n\}$ is called *compatible* with $\varepsilon \in \mathbb{A}_I$ if $|\varepsilon| = n$ and, for all $i, j \in \{1, \ldots, n\}$ that belong to the same block of Π, one has $\varepsilon_i = \varepsilon_j$. The set of all tuple partitions compatible with ε is denoted by $\mathrm{TP}(\varepsilon)$. We simply write $\mathrm{TP}(n)$ for $\mathrm{TP}(\{1, \ldots, n\})$.

3 Classification of (r, s)-Universal Products

Definition 3.1. A *universal product* is a prescription $\boxdot$ that assigns to each pair of linear functionals $\varphi_i : \mathcal{A}_i \to \mathbb{C}$ on algebras $\mathcal{A}_i$ a linear functional $\varphi_1 \boxdot \varphi_2 : \mathcal{A}_1 \sqcup \mathcal{A}_2 \to \mathbb{C}$ such that the following axioms hold:

UP1 $(\varphi_1 \boxdot \varphi_2) \circ (j_1 \amalg j_2) = (\varphi_1 \circ j_1) \boxdot (\varphi_2 \circ j_2)$ for all algebra homomorphisms $j_i : \mathcal{A}_i \to \mathcal{B}_i$, where $i \in \{1, 2\}$

UP2 $(\varphi_1 \boxdot \varphi_2) \boxdot \varphi_3 = \varphi_1 \boxdot (\varphi_2 \boxdot \varphi_3)$ for all linear functionals $\varphi_i : \mathcal{A}_i \to \mathbb{C}$, where $i \in \{1, 2, 3\}$

UP3 $(\varphi_1 \boxdot \varphi_2)(a) = \varphi_i(a)$ for all $a \in \mathcal{A}_i \subset \mathcal{A}_1 \sqcup \mathcal{A}_2$ and $i \in \{1, 2\}$

Remark. We call a pair $(\mathcal{A}, \varphi)$ consisting of an algebra $\mathcal{A}$ and a linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ an *algebraic propability space.* Denote by $\mathfrak{algQ}$ the category whose objects are the algebraic quantum probability spaces and whose morphisms are the functional preserving algebra homomorphisms. By UP1-UP3

$$\big((\mathcal{A}_1, \varphi_1), (\mathcal{A}_2, \varphi_2)\big) \mapsto (\mathcal{A}_1 \sqcup \mathcal{A}_2, \varphi_1 \boxdot \varphi_2) : \mathfrak{algQ} \times \mathfrak{algQ} \to \mathfrak{algQ}$$

is a bifunctor, which turns $\mathfrak{algQ}$ into a tensor category with inclusions in the sense of [7] (see Definition 4.4).

Example 3.1. For algebraic quantum probability spaces $(\mathcal{A}_i, \varphi_i)$, $i \in \{1, 2\}$, the well-known Boolean product $\diamond$ is given by

$$\varphi_1 \diamond \varphi_2(c_1 \cdots c_n) = \varphi_{\varepsilon_1}(c_1) \cdots \varphi_{\varepsilon_n}(c_n)$$

for $c_1 \cdots c_n \in \mathcal{A}_\varepsilon$. This is a universal product. It made early appearances, although not named this way, in the work of von Waldenfels [22] and Bozejko [4]. The theory of Boolean convolution was established in [19]. Nowadays it is an important part of non-commutative probability theory, see for example the work of Arizmendi and Hasebe [1], [2].

Example 3.2. For $r, s \in \mathbb{C}$ we define the *(r, s)-product*, denoted by $\curlywedge$, as

$$\varphi_1 \curlywedge \varphi_2 := (\varphi_1 \diamond \varphi_2) \circ \Phi$$

where $\Phi : \mathcal{A}_1 \sqcup \mathcal{A}_2 \to \mathcal{A}_1 \sqcup \mathcal{A}_2$ is the unique linear map with $\Phi(c) = c$ and

$$\Phi(c_1 \cdots c_m) := \begin{cases} r \cdot \Phi(c_1 \cdots c_k)\Phi(c_{k+1} \cdots c_m), & \text{if } \varepsilon_k < \varepsilon_{k+1} \\ s \cdot \Phi(c_1 \cdots c_k)\Phi(c_{k+1} \cdots c_m), & \text{if } \varepsilon_k > \varepsilon_{k+1} \end{cases}$$

for $c_1 \cdots c_m \in \mathcal{A}_\varepsilon$, $|\varepsilon| \geq 2$, and $c \in \mathcal{A}_i$. As we will see in Theorem 3.1 this is a universal product.

For given algebras $\mathcal{A}_i$, $\mathcal{B}_i$ and linear maps $f_i : \mathcal{A}_i \to \mathcal{B}_i$, $i \in \{1, 2\}$, we define a linear map $f_1 \sqcup f_2 : \mathcal{A}_1 \sqcup \mathcal{A}_2 \to \mathcal{B}_1 \sqcup \mathcal{B}_2$ via

$$(f_1 \sqcup f_2)(c_1 \cdots c_m) := f_{\varepsilon_1}(c_1) \cdots f_{\varepsilon_m}(c_m)$$

for $c_1 \cdots c_m \in \mathcal{A}_\varepsilon$.

Lemma 3.1. Let $\mathcal{A}_i$, $\mathcal{B}_i$ for $i \in \{1, 2\}$ be algebras. For linear maps $f_i : \mathcal{A}_i \to \mathcal{B}_i$ and linear functionals $\varphi_i : \mathcal{B}_i \to \mathbb{C}$ we have

$$(\varphi_1 \circ f_1) \diamond (\varphi_2 \circ f_2) = (\varphi_1 \diamond \varphi_2) \circ (f_1 \sqcup f_2).$$

Proof. Straightforward. □

For the next Lemma, we introduce the following notation: To indicate which algebras Φ is working on we write $\Phi_{\mathcal{A},\mathcal{B}}$ for $\Phi : \mathcal{A} \sqcup \mathcal{B} \to \mathcal{A} \sqcup \mathcal{B}$ and more concisely

$$\Phi_{1,2} := \Phi_{\mathcal{A}_1,\mathcal{A}_2}, \quad \Phi_{1|2,3} := \Phi_{\mathcal{A}_1 \sqcup \mathcal{A}_2,\mathcal{A}_3}, \quad \Phi_{1,2|3} := \Phi_{\mathcal{A}_1,\mathcal{A}_2 \sqcup \mathcal{A}_3}.$$

Furthermore, for $\varepsilon \in \mathbb{A}_p$ with $|\varepsilon| = m$, we define an *ascent* (or *up*) of ε to be any position $1 \le j < m$ such that $\varepsilon_j < \varepsilon_{j+1}$. Similarly, a *descent* (or *down*) is any position $1 \le j < m$ such that $\varepsilon_j > \varepsilon_{j+1}$. We denote the set of all ups by $\mathrm{u}(\varepsilon)$ and the set of all downs by $\mathrm{d}(\varepsilon)$. If ε has k ups and ℓ downs, it is clear that $k + \ell = m - 1$.

Lemma 3.2. The equation

$$(\mathrm{id} \sqcup \Phi_{2,3}) \circ \Phi_{1,2|3} = \Phi_{1,2,3} = (\Phi_{1,2} \sqcup \mathrm{id}) \circ \Phi_{1|2,3}$$

holds, where $\Phi_{1,2,3} : \mathcal{A}_1 \sqcup \mathcal{A}_2 \sqcup \mathcal{A}_3 \to \mathcal{A}_1 \sqcup \mathcal{A}_2 \sqcup \mathcal{A}_3$ is defined by

$$\Phi_{1,2,3}(c_1 \cdots c_m) := r^{\#\mathrm{u}(\varepsilon)} \cdot s^{\#\mathrm{d}(\varepsilon)} \cdot c_1 \cdots c_m,$$

for all $c_1 \cdots c_m \in \mathcal{A}_\varepsilon$, $\varepsilon \in \mathbb{A}_3$.

Proof. We show

$$(\mathrm{id} \sqcup \Phi_{2,3}) \circ \Phi_{1,2|3}(c_1 \cdots c_m) = \Phi_{1,2,3}(c_1 \cdots c_m)$$

by induction on the length m. For all $c \in \mathcal{A}_i$, $i \in \{1, 2, 3\}$, it holds that

$$\Phi_{1,2,3}(c) = c = (\mathrm{id} \sqcup \Phi_{2,3}) \circ \Phi_{1,2|3}(c).$$

Assume $(\mathrm{id} \sqcup \Phi_{2,3}) \circ \Phi_{1,23}(c_1 \cdots c_k) = \Phi_{1,2,3}(c_1 \cdots c_k)$ for all $c_1 \cdots c_k \in \mathcal{A}_\varepsilon$ with $k \le m$. Now let $|\varepsilon| = m + 1$ and $c_1 \cdots c_{m+1} \in \mathcal{A}_\varepsilon$. If $\varepsilon_i \ne 1$ for all $i \in \{1, \dots, m+1\}$ the equation is trivial. If $\varepsilon_{m+1} = 1$, we have $m \in \mathrm{d}(\varepsilon)$ and thus

$$\begin{aligned}
(\mathrm{id} \sqcup \Phi_{2,3}) \circ \Phi_{1,2|3}(c_1 \cdots c_{m+1}) &= (\mathrm{id} \sqcup \Phi_{2,3})\big(s \cdot \Phi_{1,2|3}(c_1 \cdots c_m)\Phi_{1,2|3}(c_m)\big) \\
&= s \cdot \Phi_{1,2,3}(c_1 \cdots c_m)\Phi_{1,2,3}(c_m) \\
&= \Phi_{1,2,3}(c_1 \cdots c_{m+1}).
\end{aligned}$$

Otherwise choose $i_0 \in \{1, \dots, m\}$ with $\varepsilon_{i_0} = 1$. Then $i_0 \in \mathrm{u}(\varepsilon)$, so

$$\begin{aligned}
(\mathrm{id} \sqcup \Phi_{2,3}) \circ \Phi_{1,2|3}(c_1 \cdots c_{m+1}) &= (\mathrm{id} \sqcup \Phi_{2,3})\big(r\Phi_{1,2|3}(c_1 \cdots c_{i_0})\Phi_{1,2|3}(c_{i_0+1} \cdots c_{m+1})\big) \\
&= r \cdot \Phi_{1,2,3}(c_1 \cdots c_{i_0})\Phi_{1,2,3}(c_{i_0+1} \cdots c_{m+1}) \\
&= \Phi_{1,2,3}(c_1 \cdots c_{m+1}).
\end{aligned}$$

The second equality can be proved analogously. □

Theorem 3.1. *The prescription* $\curlywedge$ *is a universal product.*

Proof. The only non-trivial point is to show associativity (UP2). This follows directly from Lemma 3.1, Lemma 3.2 and the associativity of the Boolean product. □

We will frequently use the following notation. Let $\varphi_1 : \mathcal{A}_1 \to \mathbb{C}, \varphi_2 : \mathcal{A}_2 \to \mathbb{C}$ be linear functionals, and $c_1 \cdots c_n \in \mathcal{A}_\varepsilon$. For a tuple $U = (i_1, \dots, i_m)$ such that all c_{i_k} belong to the same algebra $\mathcal{A}_j$ we use the shorthand notation

$$\varphi_U(c_1 \cdots c_n) := \varphi_j(c_{i_1} \cdots c_{i_m}).$$

Furthermore, for $\varepsilon \in \mathbb{A}_2$, $c_1 \cdots c_n \in \mathcal{A}_\varepsilon$ and a tuple partition $\Pi \in \mathrm{TP}(\varepsilon)$ we write

$$\varphi_\Pi(c_1 \cdots c_n) := \prod_{U \in \Pi} \varphi_U(c_1 \cdots c_n).$$

Theorem 3.2 (Ben Ghorbal and Schürmann, [3])**.** *Let $\boxdot$ be a universal product. Then there exist unique constants $t_\varepsilon(\Pi)$ for every $k \in \mathbb{N}$, $\varepsilon \in \mathbb{A}_k$ and $\Pi \in \mathrm{TP}(\varepsilon)$ such that*

$$\varphi_1 \boxdot \cdots \boxdot \varphi_k(c_1 \cdots c_n) = \sum_{\Pi \in \mathrm{TP}(\varepsilon)} t_\varepsilon(\Pi) \varphi_\Pi(c_1 \cdots c_n)$$

for all $c_1 \cdots c_n \in \mathcal{A}_\varepsilon$.

Theorem 5 of [3] deals with the case $k = 2$ and commutative universal products, but the proof relies on UP1 only, hence it applies to our more general situation. See also [16], Theorem 3.1.

In the following we call a universal product which fulfills

$$t_{(1,2)}(\{(1),(2)\}) = r \quad \text{and} \quad t_{(2,1)}(\{(1),(2)\}) = s$$

an (r,s)*-universal product.* By Theorem 3.2 every universal product is an (r,s)-universal product for unique constants $r, s \in \mathbb{C}$. If $r = s = q$ we speak of a q*-universal product.* A 1-universal product is also called a *normalized* universal product.

Observation 1. Let $\boxdot$ be an (r,s)-universal product. Then one easily checks that $\varphi_1 \boxdot^{\mathrm{op}} \varphi_2 := \varphi_2 \boxdot \varphi_1$ defines an (s,r)-universal product. For the universal coefficients $t_\varepsilon^{\boxdot}(\Pi)$ and $t_\varepsilon^{\boxdot^{\mathrm{op}}}(\Pi)$ one finds

$$t_\varepsilon^{\boxdot}(\Pi) = t_{\overline{\varepsilon}}^{\boxdot^{\mathrm{op}}}(\Pi)$$

where $\overline{\varepsilon}_k := 1$ if $\varepsilon_k = 2$ and vice versa.

For the case $q = 1$ we have:

Theorem 3.3 (Muraki, Theorem 2.2 of [16])**.** *There are exactly five normalized universal products, namely the tensor product ($\otimes$), the free product ($\diamondsuit$), the Boolean product ($\diamond$), the monotone product ($\triangleright$), and the antimonotone product ($\triangleleft$).*

This can be easily used to classify all q-universal products with $q \neq 0$.

Observation 2. Let $\boxdot$ be an (r,s)-universal product and $q \neq 0$. Then

$$\varphi_1 \boxdot_q \varphi_2 := q^{-1}(q\varphi_1) \boxdot (q\varphi_2)$$

defines an (rq, sq)-universal product, which can be checked easily. This yields a bijection between (r,s)-universal and (rq,sq)-universal products, as one has $(\boxdot_q)_{q^{-1}} = \boxdot = (\boxdot_{q^{-1}})_q$. This idea of parametrizing universal products is due to M. Bozejko.

As an immediate consequence one has

Theorem 3.4. *For every $q \neq 0$ there are exactly five q-universal products, namely $\otimes_q, \medwhitestar_q, \diamond_q, \rhd_q$ and $\lhd_q$.*

Let $\rho \in \mathrm{TP}(V)$. We define $\mathcal{A}_\rho := \mathbb{C}\langle x_i \mid i \in V\rangle$ the polynomial algebra in the non-commuting indeterminates x_i for $i \in V$ and $\varphi^{(\rho)} : \mathcal{A}_\rho \to \mathbb{C}$ by

$$\varphi^{(\rho)}(x_{i_1} \cdots x_{i_k}) := \begin{cases} 1, & (i_1, \dots, i_k) \in \rho \\ 0, & (i_1, \dots, i_k) \notin \rho. \end{cases}$$

Observation 3. Let $\varepsilon \in \mathbb{A}_p, \Pi \in \mathrm{TP}(n)$, where n is the length of ε. Set $V_i := \big\{k \in \{1, \dots, n\} \bigm| \varepsilon_k = i\big\}$ for $i = 1, \dots, p$. Then $\Pi \in \mathrm{TP}(\varepsilon)$ if and only if there are (necessarily unique) $\rho_i \in \mathrm{TP}(V_i)$ for $(i = 1, \dots, p)$ such that

$$\Pi = \bigcup_{i=1,\dots,p} \rho_i. \tag{2}$$

Proposition 3.1. *Let $\varepsilon \in \mathbb{A}_2$ with $|\varepsilon| = n$. Then we have*

$$\varphi^{(\rho_1)} \boxdot \varphi^{(\rho_2)}(x_1 \cdots x_n) = t_\varepsilon(\Pi)$$

for all $\Pi \in \mathrm{TP}(\varepsilon)$, where $\Pi = \rho_1 \cup \rho_2$ is the decomposition (2)*.*

Proof. We calculate

$$\varphi^{(\rho_1)} \boxdot \varphi^{(\rho_2)}(x_1 \cdots x_n) = \sum_{\Sigma \in \mathrm{TP}(\varepsilon)} t_\varepsilon(\Sigma) \underbrace{\varphi_\Sigma(x_1 \cdots x_n)}_{\delta_{\Sigma,\Pi}} = t_\varepsilon(\Pi).$$

□

Next, we deal with the case $r \neq s$. We will show that the (r,s)-product is the only universal product in this case (Theorem 3.6).

Lemma 3.3. Let $\boxdot$ be an (r,s)-universal product with $r \neq s$. Then

$$(\varphi_1 \boxdot \varphi_2)(c_1 c_2 c_3) = (\varphi_1 \curlywedge \varphi_2)(c_1 c_2 c_3)$$

for all linear functionals φ_1, φ_2 on algebras $\mathcal{A}_1, \mathcal{A}_2$ and all $c_1 c_2 c_3 \in \mathcal{A}_1 \sqcup \mathcal{A}_2$.

Proof. For simplicity of the formulas, we define for $\varepsilon \in \mathbb{A}_2$

$$\begin{aligned} u &:= t_{(121)}\big(\{(1),(3);(2)\}\big), & v &:= t_{(121)}\big(\{(13);(2)\}\big), & w &:= t_{(121)}\big(\{(31);(2)\}\big) \\ x &:= t_{(212)}\big(\{(2);(1),(3)\}\big), & y &:= t_{(212)}\big(\{(2);(13)\}\big), & z &:= t_{(212)}\big(\{(2);(31)\}\big) \end{aligned}$$

where the semicolons have no actual meaning, but are used to visually seperate ρ_1 and ρ_2 of the decomposition (2). We evaluate $\varphi_{1|2,3}$ and $\varphi_{1,2|3}$ on $c_1 \cdots c_n \in \mathcal{A}_\delta$ for some special $\delta \in \mathbb{A}_3$. Then we use Theorem 3.2 and the associativity of $\boxdot$ to get:

1.	$\delta = (1,3,2)$ $u + rv + sw = rs$	2.	$\delta = (2,3,1)$ $u + sv + rw = rs$	3.	$\delta = (2,1,3)$ $rs = x + ry + sz$
4.	$\delta = (3,1,2)$ $rs = x + sy + rz$	5.	$\delta = (1,2,3,1)$ $ur + uv = ur$ $v^2 = vr$ $wr + wv = wr$	6.	$\delta = (1,2,3,1)$ $xr = xr + xy$ $yr = y^2$ $zr = zr + yz$

We subtract the equation in 1. from the equation in 2. to get $v = w$ using $r \neq s$. In the same way 3. and 4. yield $y = z$. Using these in the third equation of 5. and 6. respectively we find $v = w = y = z = 0$. Finally, from 1. and 3. we get $x = u = rs$, which finishes the proof. □

Before we can prove equality on elements of arbitrary length, we have to show that the coefficents $t_\varepsilon(\Pi)$ vanish whenever Π contains a *wrong-ordered block*, that is a block $U = (i_1, \dots, i_\alpha, i_{\alpha+1}, \dots i_k)$ with $i_\alpha > i_{\alpha+1}$ for some $\alpha \in \{1, \dots, k-1\}$. This result is stated in Theorem 3.5 and prepared in a series of Lemmas. In the following we use the short hand notations

$$\varphi_{1|2,3} := \big((\varphi_1 \boxdot \varphi_2) \boxdot \varphi_3\big), \quad \varphi_{1,2|3} := \big(\varphi_1 \boxdot (\varphi_2 \boxdot \varphi_3)\big).$$

Lemma 3.4. Let $\boxdot$ be an (r,s)-universal product with $r, s \in \mathbb{C} \setminus \{0\}$ and $r \neq s$. Furthermore, we assume $\varepsilon \in \mathbb{A}_2$ and $\Pi \in \mathrm{TP}(\varepsilon)$. Let n denote the length of ε. If Π contains a block

$$U_0 = (i_1, \dots, i_k, i_{k+1}, \dots, i_l)$$

of one of the special forms

(i) $U_0 = (R, n, x, R')$, $x \in \{1, \dots, n-1\}$ or

(ii) $U_0 = (R, x, 1, R')$, $x \in \{2, \dots, n\}$

with $R = (i_1, \dots, i_{k-1})$ and $R' = (i_{k+2}, \dots, i_l)$, then $t_\varepsilon(\Pi) = 0$.

Proof. Without loss of generality assume that Π contains a block U_0 of form (i). By Observation 1 we may assume $\varepsilon_n = 2$. Let ρ_i be as in Observation 3, so $U_0 \in \rho_2$. Set $\varphi_i := \varphi^{(\rho_i)}$ $(i = 1, 2)$. We define a third functional $\varphi_3 : \mathbb{C}\langle z \rangle \to \mathbb{C}$ with $\varphi_3(z) = \frac{1}{r}$. We calculate

$$\varphi_{1|2,3}(x_1 \cdots x_n z) = r\varphi_3(z)(\varphi_1 \boxdot \varphi_2)(x_1 \cdots x_n) = t_\varepsilon(\Pi).$$

Now let $\psi_1 := \varphi_1$, $\psi_2 := \varphi_2 \boxdot \varphi_3$, $y_i := x_i$ for $i = 1, \dots, n-1$, $y_n := x_n z$. Then we find

$$\begin{aligned}
\varphi_{1,2|3}(x_1 \cdots x_n z) &= (\psi_1 \boxdot \psi_2)(y_1 \cdots y_n) \\
&= \sum_{\Sigma \in \mathrm{TP}(\varepsilon)} \psi_\Sigma(y_1 \cdots y_n) \\
&= \sum_{\Sigma \in \mathrm{TP}(\varepsilon)} \prod_{V \in \Sigma, n \notin V} \psi_V(y_1 \cdots y_n) \psi_{V_0}(y_1 \cdots y_n)
\end{aligned}$$

where $V_0 = (j_1, \ldots, j_k)$ refers to the block of Σ that contains n, say $j_\alpha = n$. We proceed to calculate

$$\begin{aligned}\psi_{V_0}(y_1 \cdots y_n) &= (\varphi_2 \boxdot \varphi_3)(y_{j_1} \cdots y_{j_{\alpha-1}} y_{j_\alpha} y_{j_{\alpha+1}} \cdots y_{j_k}) \\ &= (\varphi_2 \boxdot \varphi_3)(x_{j_1} \cdots x_{j_{\alpha-1}} x_n z x_{j_{\alpha+1}} \cdots x_{j_k}).\end{aligned}$$

The result contains the factor $\varphi_2(x_{j_1} \cdots x_{j_{\alpha-1}} x_n)$, which is immediate if $k = \alpha$, and which follows from Lemma 3.3 if $k > \alpha$. The tuple $(j_1, \ldots, j_{\alpha-1}, n)$ has the element n in common with U_0 but is clearly different from U_0, since U_0 does not end with n. Since ρ_2 is a tuple partition, we may conclude $(j_1, \ldots, j_{\alpha-1}, n) \notin \rho_2$. This shows $\varphi_2(x_{j_1} \cdots x_{j_{\alpha-1}} x_n) = 0$. By associativity we get $t_\varepsilon(\Pi) = 0$. □

Lemma 3.5. Let $\boxdot$ be an (r,s)-universal product with $r, s \in \mathbb{C} \setminus \{0\}$ and $r \neq s$. Furthermore, let $\varepsilon \in \mathbb{A}_2$ with $|\varepsilon| = 4$ and $\Pi \in \mathrm{TP}(\varepsilon)$. If Π contains a block of length 2, then $t_\varepsilon(\Pi) = 0$.

Proof. We know that the coefficient $t_\varepsilon(\Pi)$ vanishes if Π contains a wrong-ordered block by the preceding Lemma. Suppose $\varepsilon_2 = \varepsilon_4 = 2$ and $(2,4) \in \Pi$. Define $\varphi_i = \varphi^{(\rho_i)}$ for $i \in \{1,2\}$ and a third functional $\varphi_3 : \mathbb{C}\langle z \rangle \to \mathbb{C}$ with $\varphi_3(z) = 1$. We have

$$\varphi_{1,2|3}\big(x_1(zx_2)x_3x_4\big) = s\varphi_3(z)t_\varepsilon(\Pi)$$

and, using Lemma 3.3,

$$\varphi_{1|2,3}\big((x_1)z(x_2x_3x_4)\big) = rs(\varphi_1 \boxdot \varphi_2)(x_1)\varphi_3(z)(\varphi_1 \boxdot \varphi_2)(x_2x_3x_4).$$

We continue to calculate $(\varphi_1 \boxdot \varphi_2)(x_2x_3x_4) = rs\varphi_2(x_2)\varphi_1(x_3)\varphi_2(x_4)$. Since $(2,4) \in \rho_2$ and ρ_2 is a tuple partition, $(4) \notin \rho_2$. So $\varphi_2(x_4) = 0$. By associativity we get $t_\varepsilon(\Pi) = 0$. The other cases work analogously. □

Lemma 3.6. Let $\boxdot$ be an (r,s)-universal product with $r, s \in \mathbb{C} \setminus \{0\}$ and $r \neq s$. Furthermore, let $\varepsilon \in \mathbb{A}_2$ with $|\varepsilon| = 5$ and $\Pi \in \mathrm{TP}(\varepsilon)$. If Π contains the block $(4,2)$ or the block $(2,4)$ then $t_\varepsilon(\Pi) = 0$.

Proof. Let $(4,2) \in \Pi$. Without loss of generality, assume $\varepsilon_2 = \varepsilon_4 = 2$. Define $\varphi_i = \varphi^{(\rho_i)}$ for $i \in \{1,2\}$ and a third functional $\varphi_3 : \mathbb{C}\langle z \rangle \to \mathbb{C}$ with $\varphi_3(z) = 1$. We have

$$\varphi_{1,2|3}\big(x_1x_2x_3(zx_4)x_5\big) = s\varphi_3(z)t_\varepsilon(\Pi)$$

and, using Lemma 3.3,

$$\varphi_{1|2,3}\big((x_1x_2x_3)z(x_4x_5)\big) = rs(\varphi_1 \boxdot \varphi_2)(x_1x_2x_3)\varphi_3(z)(\varphi_1 \boxdot \varphi_2)(x_4x_5).$$

We continue with calculating $(\varphi_1 \boxdot \varphi_2)(x_4x_5) = s\varphi_2(x_4)\varphi_1(x_5)$. Since $(4,2) \in \rho_2$ and ρ_2 is a tuple partition, $(4) \notin \rho_2$. So $\varphi_2(x_4) = 0$. By associativity we get $t_\varepsilon(\Pi) = 0$.
Now suppose $(2,4) \in \Pi$. Then on the one hand,

$$\varphi_{1,2|3}\big(x_1x_2x_3(x_4z)x_5\big) = r\varphi_3(z)t_\varepsilon(\Pi)$$

and on the other hand,

$$\varphi_{1|2,3}\big((x_1x_2x_3x_4)z(x_5)\big) = rs(\varphi_1 \boxdot \varphi_2)(x_1x_2x_3x_4)\varphi_3(z)(\varphi_1 \boxdot \varphi_2)(x_5).$$

By Lemma 3.5 and the fact that $\varphi_2(x_4) = 0$, we can conclude

$$(\varphi_1 \boxdot \varphi_2)(x_1x_2x_3x_4) = 0.$$

Finally, associativity yields $t_\varepsilon(\Pi) = 0$. □

Theorem 3.5 (Vanishing of the wrong-ordered terms)**.** *Let $\boxdot$ be an (r,s)-universal product with $r, s \in \mathbb{C} \setminus \{0\}$ and $r \neq s$. Assume $\varepsilon \in \mathbb{A}_2$ and $\Pi \in \mathrm{TP}(\varepsilon)$. If Π contains a wrong-ordered block, then $t_\varepsilon(\Pi) = 0$.*

Proof. For $|\varepsilon| \in \{1,2\}$ this is obvious. For $|\varepsilon| \in \{3,4\}$ it follows directly from Lemma 3.4. For $|\varepsilon| = 5$ it follows from Lemma 3.4 together with Lemma 3.6.
Now let $|\varepsilon| = n \geq 5$ and $U_0 = (i_1, \ldots, i_m) \in \Pi$ be a wrong-ordered block, say $I := i_{\alpha+1} < i_\alpha =: J$. Define $U_0^1 := (i_1, \ldots, i_\alpha = J)$ and $U_0^2 := (I = i_{\alpha+1}, \ldots, i_m)$, so U_0 is split into two blocks U_0^1 and U_0^2. We put $\Pi' := \Pi \setminus \{U_0\} \cup \{U_0^1, U_0^2\}$ and observe that also $\Pi' \in \mathrm{TP}(\varepsilon)$. Now set $\varphi_1 := \varphi^{(\rho_1')}$, $\varphi_2 := \varphi^{(\rho_2')}$ and $\varphi_3 : \mathbb{C}\langle z, z'\rangle \to \mathbb{C}$ such that $\varphi_3(z'z) = 1$ and vanishes on all other monomials. We also write $\psi_1 := \varphi_1$ and $\psi_2 := \varphi_2 \boxdot \varphi_3$.
On the one hand, we have

$$\begin{aligned}\varphi_{1,2|3}\big(x_1 \cdots (zx_I) \cdots (x_Jz') \cdots x_n\big) &= \sum_{\Sigma \in \mathrm{TP}(\varepsilon)} t_\varepsilon(\Sigma)\psi_\Sigma(x_1 \cdots (zx_I) \cdots (x_Jz') \cdots x_n) \\ &= rs\; t_\varepsilon(\Pi),\end{aligned}$$

which is shown as follows: Using Lemma 3.6 and the fact that φ_3 vanishes on monomials different from $z'z$, we find $\psi_\Sigma(x_1 \cdots x_n) = 0$ when Σ does not contain a block of the form (R, J, I, R'). Suppose $\psi_\Sigma(x_1 \cdots x_n) \neq 0$ and set $V_0 := (R, J, I, R')$. It is clear that all the other blocks of $V \in \Sigma \setminus \{V_0\}$ have to be blocks of Π as well, since $\psi_V(x_1 \cdots x_n) = \varphi_V(x_1 \cdots x_n) \neq 0$. Finally,

$$\begin{aligned}\psi_{V_0}\big(x_1 \cdots (zx_I) \cdots (x_Jz') \cdots x_n\big) &= (\varphi_2 \boxdot \varphi_3)(x_Rx_Jz'zx_Ix_{R'}) \\ &= rs\; \varphi_2(x_Rx_J)\varphi_2(x_Ix_{R'}) \\ &= rs\; \delta_{(R,J),U_0^1}\delta_{(I,R'),U_0^2} \\ &= rs\; \delta_{V_0,U_0}.\end{aligned}$$

On the other hand,

$$\varphi_{1|2,3}\big((x_1 \cdots)z(x_I \cdots x_J)z'(\cdots x_n)\big) = 0$$

by Lemma 3.6.

□

Lemma 3.7. Let $\boxdot$ be an (r,s)-universal product with $r, s \in \mathbb{C} \setminus \{0\}$ and $r \neq s$. Then $\boxdot = \curlywedge$.

Proof. We show that

$$\varphi_1 \boxdot \varphi_2(c_1 \cdots c_n) = \varphi_1 \curlywedge \varphi_2(c_1 \cdots c_n) \tag{3}$$

for all $c_1 \cdots c_n \in \mathcal{A}_\varepsilon$ by induction on n. By Lemma 3.3 we know that the hypothesis holds for $n \le 3$. Now let $n \ge 3$ and suppose the (3) holds for all $k \le n$. Without loss of generality let $\varepsilon_n = 1$, then

$$\begin{aligned}
\varphi_{1|2,3}\big((x_1 \cdots x_{n-1})z(x_n)\big) &= rs\ (\varphi_1 \boxdot \varphi_2)(x_1 \cdots x_{n-1})\varphi_3(z)\varphi_1(x_n) \\
&= rs\ (\varphi_1 \curlywedge \varphi_2)(x_1 \cdots x_{n-1})\varphi_3(z)\varphi_1(x_n) \\
&= r\varphi_3(z)\ (\varphi_1 \curlywedge \varphi_2)(x_1 \cdots x_n)
\end{aligned}$$

and for $\psi_1 := \varphi_1$ and $\psi_2 := \varphi_2 \boxdot \varphi_3$ we calculate

$$\begin{aligned}
(\psi_1 \boxdot \psi_2)\big(x_1 \cdots (x_{n-1}z)x_n\big) &= \sum_{\Sigma \in \mathrm{TP}(\varepsilon)} t_\varepsilon(\Sigma)\psi_\Sigma\big(x_1 \cdots (x_{n-1}z)x_n\big) \\
&= \sum_{\Sigma \in \mathrm{TP}(\varepsilon)} t_\varepsilon(\Sigma) r\varphi_3(z)\varphi_\Sigma(x_1 \cdots x_n) \\
&= r\varphi_3(z)\ (\varphi_1 \boxdot \varphi_2)(x_1 \cdots x_n),
\end{aligned}$$

where the difficult step is from the first to the second line: By use of Theorem 3.5, $\psi_\Sigma(x_1 \cdots x_{n-1}zx_n) \neq 0$ implies that the block $V_0 \in \Sigma$ which contains the element $n-1$ ends with $n-1$, so

$$\begin{aligned}
\psi_{V_0}(x_1 \cdots x_{n-1}zx_n) &= \varphi_2 \boxdot \varphi_3(\cdots x_{n-1}z) \\
&= r\varphi_2(\cdots x_{n-1})\varphi_3(z) \\
&= r\varphi_3(z)\varphi_{V_0}(x_1 \cdots x_n).
\end{aligned}$$

□

Lemma 3.8. Let $\boxdot$ be an (r,s)-universal product. For the cases

(*i*) $r = 0$ and $s \neq 0$

(*ii*) $s = 0$ and $r \neq 0$

the product $\boxdot$ is unique and fullfills

$$\varphi_1 \boxdot \varphi_2(c_1 \cdots c_n) = 0$$

for all $c_1 \cdots c_n \in \mathcal{A}_1 \sqcup \mathcal{A}_2$ with $n \ge 3$.

Proof. By Observation 1, the statement in case (i) is equivalent to the statement in case (ii). We consider situation (ii). Let $\varepsilon \in \mathbb{A}_2$, $|\varepsilon| =: n \ge 3$ and $\Pi = \rho_1 \cup \rho_2 \in \mathrm{TP}(\varepsilon)$. We define $\varphi_i := \varphi^{(\rho_i)}$ for $i \in \{1,2\}$ and $\varphi_3 : \mathbb{C}\langle z\rangle \to \mathbb{C}$, with $\varphi_3(z) := 1$.

Part 1. Assume there is a block $V_0 = (i_1, \ldots, i_k) \in \rho_2$ with $i_k \neq n$. As earlier put $\psi_1 := \varphi_1$ and $\psi_2 := \varphi_2 \boxdot \varphi_3$. Then we can calculate

$$\begin{aligned}
\varphi_{1,2|3}(x_1 \cdots x_{i_k} z x_{i_k+1} \cdots x_n) &= \sum_{\Sigma \in \mathrm{TP}(\varepsilon)} t_\varepsilon(\Sigma)\psi_\Sigma(x_1 \cdots (x_{i_k}z)x_{i_k+1} \cdots x_n) \\
&= r \cdot t_\varepsilon(\Pi).
\end{aligned}$$

To see this, first observe that $\psi_\Sigma\big(x_1\cdots(x_{i_k}z)x_{i_k+1}\cdots x_n\big)$ vanishes when the block of $V_0'\in\Sigma$ which contains the element i_k does not end with i_k, because $\psi_{V_0'}(x_1\cdots(x_{i_k}z)x_{i_k+1}\cdots x_n)=\varphi_2\boxdot\varphi_3(\cdots x_{i_k}zx_j\cdots)=0$ by Lemma 3.3. Then by the definition of φ_1 and φ_2 we get $\psi_\Sigma(x_1\cdots(x_{i_k}z)x_{i_k+1}\cdots x_n)=r\delta_{\Sigma,\Pi}$.
Using Lemma 3.3 again, we see that $\varphi_{1|2,3}(x_1\cdots x_{i_k}zx_{i_k+1}\cdots x_n)=0$. From associativity and $r\neq 0$ we conclude $t_\varepsilon(\Pi)=0$.

Part 2. It remains to show that the coefficients also vanish if ρ_2 consists of only one block of the form $V_0=(R,n)$. We prove this by induction over the length n. For $n=3$ this follows with Lemma 3.3. Assume that the hypothesis holds for $3\le k\le n$. Then by proofpart 1, we have $t_\varepsilon(\Pi)=0$ for all ε with $3\le|\varepsilon|\le n$. Let $\Pi=\rho_1\cup\rho_2\in\mathrm{TP}(\varepsilon)$ with $|\varepsilon|=n+1$ and $\rho_2=\{V_0=(R,n+1)\}$. We define $\varphi_1:=\varphi^{(\rho_1)}$ and $\varphi_2:=\varphi^{\rho_2'}$, where $\rho_2':=\{V_0'=(R)\}$, that is ρ_2' is obtained from ρ_2 by leaving out the element $n+1$. Let $\varphi_3(z)=1$ again. We find

$$\varphi_{1|2,3}(x_1\cdots x_nz)=r\varphi_1\boxdot\varphi_2(x_1\cdots x_n)\varphi_3(z)=0$$

because of the assumption.
On the other hand, we obtain the equation

$$\varphi_{1,2|3}(x_1\cdots x_nz)=r\cdot t_\varepsilon(\Pi)+t_\varepsilon(\Sigma)$$

where $\Sigma=\rho_1\cup\rho_2'\cup\{(n+1)\}$. But $t_\varepsilon(\Sigma)=0$ by proofpart 1, so we get $0=t_\varepsilon(\Pi)$.

$\square$

To sum up, we have shown:

Theorem 3.6. *Let $\boxdot$ be an (r,s)-universal product. If $r\neq s$ then $\boxdot=\curlywedge$.*

Thus, by Theorem 3.4 and 3.6, we have classified all universal products except the 0-universal ones. We can do calculations which show that a 0-universal product fulfills $t_\varepsilon(\Pi)=0$ for all $\Pi\in\mathrm{TP}(\varepsilon)$ whenever $2\le|\varepsilon|\le 5$. But it is an open question, even in the case of commutative universal products, whether the *degenerate product*, that is the universal product with all $t_\varepsilon(\Pi)=0$ for $|\varepsilon|\ge 2$, is the only 0-universal product or not.

4 Reduction to tensor independence

Our aim is to embed each algebraic probability space $(\mathcal{A},\varphi)$ into a unital algebraic probability space $(\widehat{\mathcal{A}},\widehat{\varphi})$ such that there are functional preserving unital algebra homomorphisms

$$\mathfrak{E}_{\mathcal{A},\mathcal{B}}:\widehat{\mathcal{A}\sqcup\mathcal{B}}\to\widehat{\mathcal{A}}\otimes\widehat{\mathcal{B}}$$

which fulfill natural compatibility conditions that will allow us to construct $\otimes$-independent qunatum random variables from (r,s)-independent quantum random variables. What we need is exactly what is called a cotensor functor in the literature, so we briefly recall the definitions of tensor categories and cotensor functors.

Definition 4.1. A *tensor category* is a category $\mathfrak{K}$ together with a bifunctor $\boxtimes : \mathfrak{K} \times \mathfrak{K} \to \mathfrak{K}$ which

- is associative under the natural isomorphism
$$\alpha_{A,B,C} : A \boxtimes (B \boxtimes C) \xrightarrow{\cong} (A \boxtimes B) \boxtimes C$$
- has a unit object $E \in \mathrm{obj}(\mathfrak{K})$ acting as left and right identity under the natural isomorphisms
$$l_A : E \boxtimes A \xrightarrow{\cong} A, \quad r_A : A \boxtimes E \xrightarrow{\cong} A$$

such that the diagrams

$$\begin{tikzcd}
 & (A \boxtimes B) \boxtimes (C \boxtimes D) \arrow[dr, "\alpha_{A\boxtimes B,C,D}"] & \\
A \boxtimes \big(B \boxtimes (C \boxtimes D)\big) \arrow[ur, "\alpha_{A,B,C\boxtimes D}"] \arrow[d, "\mathrm{id}_A \boxtimes \alpha_{B,C,D}"'] \arrow[rr, phantom, "\circlearrowleft"] & & \big((A \boxtimes B) \boxtimes C\big) \boxtimes D \\
A \boxtimes \big((B \boxtimes C) \boxtimes D\big) \arrow[rr, "\alpha_{A,B\boxtimes C,D}"] & & \big(A \boxtimes (B \boxtimes C)\big) \boxtimes D \arrow[u, "\alpha_{A,B,C} \boxtimes \mathrm{id}_D"']
\end{tikzcd}$$

$$\begin{tikzcd}
A \boxtimes (E \boxtimes C) \arrow[rr, "\alpha_{A,E,C}"] \arrow[dr, "\mathrm{id}_A \boxtimes l_C"'] & \arrow[d, phantom, "\circlearrowleft"] & (A \boxtimes E) \boxtimes C \arrow[dl, "r_A \boxtimes \mathrm{id}_C"] \\
 & A \boxtimes C &
\end{tikzcd}$$

commute for all $A, B, C \in \mathrm{obj}(\mathfrak{K})$. A tensor category will be usually be denoted by $(\mathfrak{K}, \boxtimes)$.

Definition 4.2. A *comonoid* (C, Δ, δ) in $(\mathfrak{K}, \boxtimes)$ is an object $C \in \mathrm{obj}(\mathfrak{K})$ with morphisms

- $\Delta : C \to C \boxtimes C$ (comultiplication)
- $\delta : C \to E$ (counit)

such that the following diagrams commute:

$$\begin{tikzcd}
\mathcal{C} \boxtimes \mathcal{C} \arrow[d, "\mathrm{id}_{\mathcal{C}} \boxtimes \Delta"'] & \mathcal{C} \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, phantom, "\circlearrowleft"] & \mathcal{C} \boxtimes \mathcal{C} \arrow[d, "\Delta \boxtimes \mathrm{id}_{\mathcal{C}}"] \\
\mathcal{C} \boxtimes (\mathcal{C} \boxtimes \mathcal{C}) \arrow[rr, "\alpha_{\mathcal{C},\mathcal{C},\mathcal{C}}"] & & (\mathcal{C} \boxtimes \mathcal{C}) \boxtimes \mathcal{C}
\end{tikzcd}$$

$$\begin{tikzcd}
E \boxtimes \mathcal{C} \arrow[ddr, "l_{\mathcal{C}}"'] & \mathcal{C} \boxtimes \mathcal{C} \arrow[l, "\delta \boxtimes \mathrm{id}_{\mathcal{C}}"'] \arrow[r, "\mathrm{id}_{\mathcal{C}} \boxtimes \delta"] & \mathcal{C} \boxtimes E \arrow[ddl, "r_{\mathcal{C}}"] \\
\arrow[r, phantom, "\circlearrowleft"] & \mathcal{C} \arrow[u, "\Delta"'] \arrow[d, "\mathrm{id}"] \arrow[r, phantom, "\circlearrowleft"] & \\
 & \mathcal{C} &
\end{tikzcd}$$

We refer to any comonoid in the tensor category $(\mathfrak{alg}_{\mathbb{1}}, \otimes)$ as a *bialgebra* and in the tensor category $(\mathfrak{alg}, \sqcup)$ it is said to be a *dual semigroup*. This definition agrees with Voiculescu's definition of a dual semigroup as in [21].

Definition 4.3. Given tensor categories $(\mathfrak{K}, \boxtimes)$ and $(\mathfrak{K}', \boxtimes')$ with unit objects E and E' respectively, a *cotensor functor* is a triple $(\mathcal{F}, g_0, \mathfrak{T})$ consisting of

- a functor $\mathcal{F} : \mathfrak{K} \to \mathfrak{K}'$
- a morphism $g_0 : \mathcal{F}(E) \to E'$
- a natural transformation $\mathfrak{T} : \mathcal{F}(\cdot \boxtimes \cdot) \Rightarrow \mathcal{F}(\cdot) \boxtimes' \mathcal{F}(\cdot)$

such that the diagrams

$$\begin{tikzcd}
\mathcal{F}\big(A \boxtimes (B \boxtimes C)\big) \arrow[r, "\mathcal{F}(\alpha_{A,B,C})"] \arrow[d, "\mathfrak{T}_{A,B\boxtimes C}"'] & \mathcal{F}\big((A \boxtimes B) \boxtimes C\big) \arrow[d, "\mathfrak{T}_{A\boxtimes B,C}"] \\
\mathcal{F}(A) \boxtimes' \mathcal{F}(B \boxtimes C) \arrow[r, phantom, "\circlearrowleft"] \arrow[d, "\mathrm{id}_{\mathcal{F}(A)} \boxtimes' \mathfrak{T}_{B,C}"'] & \mathcal{F}(A \boxtimes B) \boxtimes' \mathcal{F}(C) \arrow[d, "\mathfrak{T}_{A,B} \boxtimes' \mathrm{id}_{\mathcal{F}(C)}"] \\
\mathcal{F}(A) \boxtimes' \big(\mathcal{F}(B) \boxtimes' \mathcal{F}(C)\big) \arrow[r, "\alpha'_{\mathcal{F}(A),\mathcal{F}(B),\mathcal{F}(C)}"] & \big(\mathcal{F}(A) \boxtimes' \mathcal{F}(B)\big) \boxtimes' \mathcal{F}(C)
\end{tikzcd} \tag{4}$$

$$\begin{tikzcd}
\mathcal{F}(B \boxtimes E) \arrow[r, "\mathfrak{T}_{B,E}"] \arrow[d, "\mathcal{F}(r_B)"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{F}(B) \boxtimes' \mathcal{F}(E) \arrow[d, "\mathrm{id}_{\mathcal{F}(B)} \boxtimes' g_0"] \\
\mathcal{F}(B) & \mathcal{F}(B) \boxtimes' E' \arrow[l, "r'_{\mathcal{F}(B)}"']
\end{tikzcd} \tag{5}$$

$$\begin{tikzcd}
\mathcal{F}(E \boxtimes B) \arrow[r, "\mathfrak{T}_{E,B}"] \arrow[d, "\mathcal{F}(l_B)"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{F}(E) \boxtimes' \mathcal{F}(B) \arrow[d, "g_0 \boxtimes' \mathrm{id}_{\mathcal{F}(B)}"] \\
\mathcal{F}(B) & E' \boxtimes' \mathcal{F}(B) \arrow[l, "l'_{\mathcal{F}(B)}"']
\end{tikzcd} \tag{6}$$

commute for all $A, B, C \in \mathrm{obj}(\mathfrak{K})$.

By $\mathcal{P}_j : \mathfrak{K} \times \mathfrak{K} \to \mathfrak{K}$ we denote the projection functors on the first respectively second component, that is $\mathcal{P}_j(\mathcal{A}_1, \mathcal{A}_2) = \mathcal{A}_j$, $\mathcal{P}_j(f_1, f_2) = f_j$ for any objects $\mathcal{A}_1, \mathcal{A}_2$ and any morphisms f_1, f_2.

Definition 4.4. A *tensor category with inclusions* is a quadrupel $(\mathfrak{K}, \boxtimes, \iota_1, \iota_2)$ with

- $(\mathfrak{K}, \boxtimes)$ tensor category,
- $\iota_j : \mathcal{P}_j \Rightarrow \boxtimes$ natural transformations for $j \in \{1, 2\}$,

that is for $B_1, B_2 \in \mathrm{obj}(\mathfrak{K})$ there exist $\iota_{B_{1/2}} \in \mathrm{mor}(B_{1/2}, B_1 \boxtimes B_2)$ such that

$$\begin{tikzcd}
A_1 \arrow[r, "\iota_{A_1}"] \arrow[d, "f_1"'] \arrow[dr, phantom, "\circlearrowleft"] & A_1 \boxtimes A_2 \arrow[d, "f_1 \boxtimes f_2"] \arrow[dr, phantom, "\circlearrowleft"] & A_2 \arrow[l, "\iota_{A_2}"'] \arrow[d, "f_2"] \\
B_1 \arrow[r, "\iota_{B_2}"] & B_1 \boxtimes B_2 & B_2 \arrow[l, "\iota_{B_1}"']
\end{tikzcd}$$

for all $f_{1/2} \in \mathrm{mor}(A_{1/2}, B_{1/2})$.

Clearly, $(\mathfrak{algQ}, \curlywedge)$ and $(\mathfrak{algQ}_{\mathbb{1}}, \otimes)$ are tensor categories with inclusions, where the inclusions are the canonical inclusions of algebras into their free product respectively unital algebras into their tensor product.

Definition 4.5. Let $(\mathfrak{K}_1, \boxtimes, \iota_1, \iota_2)$ and $(\mathfrak{K}_2, \boxtimes', \iota_1', \iota_2')$ be tensor categories with inclusions and $\mathcal{F}_j : \mathfrak{K}_j \to \mathfrak{K}$ functors to some category $\mathfrak{K}$. A *reduction of an independence* is a pair $(\mathcal{R}, \mathfrak{u})$ consisting of

- $\mathcal{R} : (\mathfrak{K}_1, \boxtimes) \to (\mathfrak{K}_2, \boxtimes')$ cotensor functor
- $\mathfrak{u} : \mathcal{F}_1 \Rightarrow \mathcal{F}_2 \circ \mathcal{R}$ natural transformation

We are only interested in the special case of a reduction from $(\mathfrak{algQ}, \curlywedge)$ to $(\mathfrak{algQ}_{\mathbb{1}}, \otimes)$ where $\mathcal{F}_1$ is the identity functor on $\mathfrak{algQ}$, $\mathcal{F}_2 : \mathfrak{algQ}_{\mathbb{1}} \to \mathfrak{algQ}$ is the forgetful functor and u consists of embeddings as described in the beginning of this section. The construction we are going to describe now is inspired by those in [7], especially the reduction of the Boolean independence. A main difference is that we are forced to work with a bigger semigroup, which consists of two non-identity elements p_1 and p_2 instead of one.
Consider the set $M = \{p_1, p_2, \mathbb{1}\}$ with unit element $\mathbb{1}$ and multiplication given by

$$p_i p_j = p_i, \quad i, j \in \{1, 2\}$$

The semigroup algebra $\mathbb{C}M$ becomes a bialgebra with group-like comultiplication

$$\Delta : \mathbb{C}M \to \mathbb{C}M \otimes \mathbb{C}M, \quad \Delta(m) = m \otimes m$$

and counit

$$\delta : \mathbb{C}M \to \mathbb{C}, \quad \delta(m) = 1$$

for $m \in M$. It is easy to check that these maps indeed fulfill the counit property and the coassociativity.
We assign to a quantum probability space $(\mathcal{A}, \varphi)$ the quantum probability space $(\widehat{\mathcal{A}}, \widehat{\varphi})$, where

$$\widehat{\mathcal{A}} := \widetilde{\mathcal{A}} \sqcup_{\mathbb{1}} \mathbb{C}M$$

and the linear functional $\widehat{\varphi} : \widehat{\mathcal{A}} \to \mathbb{C}$ is defined by

$$\widehat{\varphi}\left(p_{i_1}^{\alpha} c_1 p_{i_2} \cdots p_{i_n} c_n p_{i_{n+1}}^{\omega}\right) = r^{\#\{c_i p_1\}} s^{\#\{c_i p_2\}} \prod_{k=1}^{n} \varphi(c_k), \quad \widehat{\varphi}(p_j) = 1$$

for all $k \in \{1, \ldots, n\}$, $\alpha, \omega \in \{0, 1\}$, $j \in \{1, 2\}$ and $c_1, \ldots, c_n \in \mathcal{A}$.

Lemma 4.1. Let $x_1 \cdots x_n \in \widetilde{\mathcal{A}} \sqcup_{\mathbb{1}} \mathbb{C}M$ with $x_i \in \mathcal{A} \cup \{p_1, p_2\}$. Then

$$\widehat{\varphi}(x_1 \cdots x_n x_{n+1}) = \begin{cases} \widehat{\varphi}(x_1 \cdots x_n)\widehat{\varphi}(x_{n+1}) & x_n \in \{p_1, p_2\} \\ \widehat{\varphi}(x_1 \cdots x_n) t_i & x_n \in \mathcal{A} \text{ and } x_{n+1} = p_i \end{cases}$$

where $t_1 := r, t_2 := s$.

Proof. Immediate from the definition of $\widehat{\varphi}$. □

For an algebra homomorphism $f : \mathcal{A} \to \mathcal{B}$ we define $\widehat{f} := \widetilde{f} \sqcup_{\mathbb{1}} \mathrm{id}_{\mathbb{C}M} : \widehat{\mathcal{A}} \to \widehat{\mathcal{B}}$. We have:

Proposition 4.1. *The prescription given by*

$$\widehat{\cdot} : \begin{cases} (\mathcal{A}, \varphi) \in \mathrm{obj}(\mathfrak{algQ}) & \mapsto (\widehat{\mathcal{A}}, \widehat{\varphi}) \in \mathrm{obj}(\mathfrak{algQ}_{\mathbb{1}}) \\ f \in \mathrm{mor}(\mathfrak{algQ}) & \mapsto \widehat{f} \in \mathrm{mor}(\mathfrak{algQ}_{\mathbb{1}}) \end{cases}$$

is a functor from the category $\mathfrak{algQ}$ *to the category* $\mathfrak{algQ}_{\mathbb{1}}$.

Proof. Let $(\mathcal{A}, \varphi)$ be an object in $\mathfrak{algQ}$. Immediately, $\widehat{A} = \widetilde{\mathcal{A}} \sqcup_{\mathbb{1}} \mathbb{C}M$ is a unital algebra. Choose a morphism $f : (\mathcal{A}_1, \varphi_1) \to (\mathcal{A}_2, \varphi_2)$ in $\mathrm{mor}(\mathfrak{algQ})$, that is an algebra homomorphism which preserves linear functionals. Then $\widehat{f} = \widetilde{f} \sqcup_{\mathbb{1}} \mathrm{id}_{\mathbb{C}M}$ is a unital algebra homomorphism by definition. It remains to prove that

$$\widehat{\varphi_2} \circ \widehat{f} = \widehat{\varphi_1}$$

holds. We calculate

$$\begin{aligned} \widehat{\varphi_2} \circ \widehat{f}(p_{i_1}^{\alpha} c_1 p_{i_2} \cdots p_{i_m} c_m p_{i_{m+1}}^{\omega}) &= \widehat{\varphi_2}\big(p_{i_1}^{\alpha} f(c_1) p_{i_2} \cdots p_{i_m} f(c_m) p_{i_{m+1}}^{\omega}\big) \\ &= r^{\#\{f(c_i)p_1\}} s^{\#\{f(c_i)p_2\}} \prod_{k=1}^{m} \underbrace{(\varphi_2 \circ f)}_{=\varphi_1}(c_k) \\ &= \widehat{\varphi_1}(p_{i_1}^{\alpha} c_1 p_{i_2} \cdots p_{i_m} c_m p_{i_{m+1}}^{\omega}). \end{aligned}$$

Finally, we have to check that the prescription above preserves the composition of morphisms. For $f, g \in \mathrm{mor}(\mathfrak{algQ})$ we get

$$\widehat{f \circ g} = \widetilde{f \circ g} \sqcup_{\mathbb{1}} \mathrm{id}_{\mathbb{C}M} = (\widetilde{f} \circ \widetilde{g}) \sqcup_{\mathbb{1}} \mathrm{id}_{\mathbb{C}M} = (\widetilde{f} \sqcup_{\mathbb{1}} \mathrm{id}_{\mathbb{C}M}) \circ (\widetilde{g} \sqcup_{\mathbb{1}} \mathrm{id}_{\mathbb{C}M}) = \widehat{f} \circ \widehat{g}$$

and the assertion follows. □

Define

$$\mathfrak{E}_{\mathcal{A},\mathcal{B}} : \widehat{\mathcal{A} \sqcup \mathcal{B}} \to \widehat{\mathcal{A}} \otimes \widehat{\mathcal{B}}$$

as the unique unital algebra homomohrphism with

$$\mathfrak{E}_{\mathcal{A},\mathcal{B}}(a) = a \otimes p_2, \quad \mathfrak{E}_{\mathcal{A},\mathcal{B}}(b) = p_1 \otimes b \quad \text{and} \quad \mathfrak{E}_{\mathcal{A},\mathcal{B}}(p_i) = p_i \otimes p_i$$

for all $a \in \mathcal{A}, b \in \mathcal{B}, i \in \{1, 2\}$. Identifying $\widehat{\mathcal{A} \sqcup \mathcal{B}} \cong \widetilde{\mathcal{A}} \sqcup_{\mathbb{1}} \widetilde{\mathcal{B}} \sqcup_{\mathbb{1}} \mathbb{C}M$ this means

$$\mathfrak{E}_{\mathcal{A},\mathcal{B}} = \widetilde{i_1} \sqcup_{\mathbb{1}} \widetilde{i_2} \sqcup_{\mathbb{1}} \Delta$$

where $i_1 : \mathcal{A} \to \widehat{\mathcal{A}} \boxtimes \widehat{\mathcal{B}}, i_1(a) = a \otimes p_2$ and $i_2 : \widehat{\mathcal{A}} \boxtimes \widehat{\mathcal{B}}, i_2(b) = p_1 \otimes b$ and Δ is the grouplike comultiplication on $\mathbb{C}M$. This shows that $\mathfrak{E}_{\mathcal{A},\mathcal{B}}$ is indeed a well-defined unital algebra homomorphism.
Let $\curlywedge$ denote the (r,s)-product of linear functionals as well as the corresponding bifunctor $\curlywedge : \mathfrak{alg}\mathfrak{Q} \times \mathfrak{alg}\mathfrak{Q} \to \mathfrak{alg}\mathfrak{Q}$ with

$$\begin{aligned}\big((\mathcal{A}_1,\varphi_1),(\mathcal{A}_2,\varphi_2)\big) &\mapsto (\mathcal{A}_1 \sqcup \mathcal{A}_2, \varphi_1 \curlywedge \varphi_2)\\ (f_1,f_2) &\mapsto f_1 \amalg f_2.\end{aligned}$$

Proposition 4.2. *The algebra homomorphisms $\mathfrak{E}_{\mathcal{A},\mathcal{B}}$ are morphisms in the category $\mathfrak{alg}\mathfrak{Q}$ and form a natural transformation*

$$\mathfrak{E} : \widehat{\cdot \curlywedge \cdot} \Rightarrow \widehat{\cdot} \otimes \widehat{\cdot}.$$

Proof. The proof will be divided into two steps. We start with the observation that the $\mathfrak{E}_{\mathcal{A},\mathcal{B}} : \widehat{\mathcal{Q}_1 \curlywedge \mathcal{Q}_2} \to \widehat{\mathcal{Q}_1} \otimes \widehat{\mathcal{Q}_2}$ for some $\mathcal{Q}_1 = (\mathcal{A},\varphi), \mathcal{Q}_2 = (\mathcal{B},\psi) \in \mathrm{obj}(\mathfrak{alg}\mathfrak{Q})$ are morphisms, that is algebra homomorphisms which preserve the linear functionals. By definition the $\mathfrak{E}_{\mathcal{A},\mathcal{B}}$ are algebra homomorphisms. So it is sufficient to show

$$(\widehat{\varphi} \otimes \widehat{\psi}) \circ \varepsilon_{\mathcal{A},\mathcal{B}} = \widehat{\varphi \curlywedge \psi}. \tag{7}$$

In a second part of the proof we check the naturality of these morphisms.

Part 1. The algebra $\widehat{\mathcal{A} \sqcup \mathcal{B}} = \widetilde{\mathcal{A}} \sqcup_{\mathbb{1}} \widetilde{\mathcal{B}} \sqcup_{\mathbb{1}} \mathbb{C}M$ is spanned by elements of the form $x_1 \cdots x_n$ with $x_i \in \mathcal{A} \cup \mathcal{B} \cup \{p_1,p_2\}$. We show that both sides of (7) agree on elements of this form by induction on n. For $n = 0$ this is obvious. For $n = 1$ we have

$$(\widehat{\varphi} \otimes \widehat{\psi})(\varepsilon_{\mathcal{A},\mathcal{B}}(x)) = \begin{cases} \widehat{\varphi}(x)\widehat{\psi}(p_2) = \varphi(x) = \widehat{\varphi \curlywedge \psi}(x), & x \in \mathcal{A}\\ \widehat{\varphi}(p_1)\widehat{\psi}(x) = \psi(x) = \widehat{\varphi \curlywedge \psi}(x), & x \in \mathcal{B}\\ \widehat{\varphi}(p_i)\widehat{\psi}(p_i) = 1 = \widehat{\varphi \curlywedge \psi}(p_i), & x \in \{p_1,p_2\}\end{cases}$$

Now consider an element $x_1 \cdots x_{n+1}$. We may assume that x_n and x_{n+1} belong to different terms of $\mathcal{A} \cup \mathcal{B} \cup \{p_1,p_2\}$, otherwise we could define $x_n' := x_n x_{n+1}$ and get a shorter element. We write $\varepsilon_{\mathcal{A},\mathcal{B}}(x_i) = y_i \otimes z_i$ with $y_i, z_i \in \mathcal{A} \cup \mathcal{B} \cup \{p_1,p_2\}$. We have

$$(\widehat{\varphi} \otimes \widehat{\psi})\big(\varepsilon_{\mathcal{A},\mathcal{B}}(x_1 \cdots x_{n+1})\big) = \widehat{\varphi}(y_1 \cdots y_{n+1})\widehat{\psi}(z_1 \cdots z_{n+1})$$

The further calculation depends on where x_n is from.
Case $x_n = p_i$: Then $y_n = z_n = p_i$ and by Lemma 4.1 and

$$\begin{aligned}\widehat{\varphi}(y_1 \cdots y_{n+1})\widehat{\psi}(z_1 \cdots z_{n+1}) &= \widehat{\varphi}(y_1 \cdots y_n)\widehat{\varphi}(y_{n+1})\widehat{\psi}(z_1 \cdots z_n)\widehat{\psi}(z_{n+1})\\ &= \widehat{\varphi \curlywedge \psi}(x_1 \cdots x_n)\widehat{\varphi \curlywedge \psi}(x_{n+1})\\ &= \widehat{\varphi \curlywedge \psi}(x_1 \cdots x_{n+1})\end{aligned}$$

Case $x_n \in \mathcal{A}$: Then $y_n = x_n \in \mathcal{A}, z_n = p_2$ and $x_{n+1} \notin \mathcal{A}$, so $y_{n+1} = p_i \in \{p_1, p_2\}$. We get

$$\begin{aligned}
\widehat{\varphi}(y_1 \cdots y_{n+1})\widehat{\psi}(z_1 \cdots z_{n+1}) &= \widehat{\varphi}(y_1 \cdots y_n)\widehat{\varphi}(y_{n+1})t_i\widehat{\psi}(z_1 \cdots z_n)\widehat{\psi}(z_{n+1}) \\
&= t_i\widehat{\varphi \curlywedge \psi}(x_1 \cdots x_n)\widehat{\varphi \curlywedge \psi}(x_{n+1}) \\
&= *
\end{aligned}$$

a) If $x_{n+1} = p_i$, then $* = \widehat{\varphi \curlywedge \psi}(x_1 \cdots x_{n+1})$.

b) If $x_{n+1} \in \mathcal{B}$, then $y_{n+1} = p_1$. Let $x_k \in \{p_1, p_2\}$ and $x_{k+1}, \ldots, x_{n+1} \in \mathcal{A} \cup \mathcal{B}$. Hence, we have

$$\begin{aligned}
* &= r\widehat{\varphi \curlywedge \psi}(x_1 \cdots x_k)\varphi \curlywedge \psi(x_{k+1} \cdots x_n)\varphi \curlywedge \psi(x_{n+1}) \\
&= \widehat{\varphi \curlywedge \psi}(x_1 \cdots x_k)\varphi \curlywedge \psi(x_{k+1} \cdots x_{n+1}) \\
&= \widehat{\varphi \curlywedge \psi}(x_1 \cdots x_{n+1}).
\end{aligned}$$

Part 2. For proving the commutativity of the the diagram

$$\begin{tikzcd}
\widehat{\mathcal{A} \sqcup \mathcal{B}} \arrow[r, "\mathfrak{E}_{\mathcal{A},\mathcal{B}}"] \arrow[d, "\widehat{f \sqcup g}"'] & \widehat{\mathcal{A}} \otimes \widehat{\mathcal{B}} \arrow[d, "\widehat{f} \otimes \widehat{g}"] \\
\widehat{\mathcal{C} \sqcup \mathcal{D}} & \widehat{\mathcal{C}} \otimes \widehat{\mathcal{D}} \arrow[l, "\mathfrak{E}_{\mathcal{C},\mathcal{D}}"']
\end{tikzcd}$$

we have to check

$$(\widehat{f} \otimes \widehat{g}) \circ \mathfrak{E}_{\mathcal{A},\mathcal{B}} = \mathfrak{E}_{\mathcal{C},\mathcal{D}} \circ \widehat{f \sqcup g}.$$

All morphisms in the diagramm above are unital algebra homomorphisms, so it is sufficient to calculate on genererating elements $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $p_i \in \mathbb{C}M$, $i \in \{1, 2\}$. We get for $a \in \mathcal{A}$ (similarily for $b \in \mathcal{B}$)

$$\begin{aligned}
(\widehat{f} \otimes \widehat{g}) \circ \mathfrak{E}_{\mathcal{A},\mathcal{B}}(a) &= (\widehat{f} \otimes \widehat{g})(a \otimes p_2) = f(a) \otimes p_2 \\
\mathfrak{E}_{\mathcal{C},\mathcal{D}} \circ \widehat{f \sqcup g}(a) &= \mathfrak{E}_{\mathcal{C},\mathcal{D}}\big(f(a)\big) = f(a) \otimes p_2
\end{aligned}$$

and for $p_i \in \mathbb{C}M$, $i \in \{1, 2\}$,

$$\begin{aligned}
(\widehat{f} \otimes \widehat{g}) \circ \mathfrak{E}_{\mathcal{A},\mathcal{B}}(p_i) &= \widehat{f} \otimes \widehat{g}(p_i \otimes p_i) = p_i \otimes p_i \\
\mathfrak{E}_{\mathcal{C},\mathcal{D}} \circ \widehat{f \sqcup g}(p_i) &= \mathfrak{E}_{\mathcal{C},\mathcal{D}}(p_i) = p_i \otimes p_i.
\end{aligned}$$

□

Theorem 4.1. *The triple* $(\widehat{\cdot}, g_0, \mathfrak{E})$ *is a cotensor functor and a reduction with respect to the canonical inclusions* $\mathcal{A} \subset \widehat{\mathcal{A}}$.

Proof. From Proposition 4.1 we have $\widehat{\cdot} : (\mathfrak{alg}\mathfrak{Q}, \boxtimes) \to (\mathfrak{alg}\mathfrak{Q}_{\mathbb{1}}, \otimes)$ is a functor. By Proposition 4.2 we conclude that $\mathfrak{E} : \widehat{\cdot \boxtimes \cdot} \Rightarrow \widehat{\cdot} \boxtimes \widehat{\cdot}$ is a natural transformation. Since $\widehat{\{0\}} = \mathbb{C}M$ holds, we see that

$$g_0 : \mathbb{C}M \to \mathbb{C}, \quad g_0(m) = 1$$

for $m \in \mathbb{C}M$ is multiplicative on the semigroup M. Hence, it extends to an algebra homomorphism.
It remains to show the properties (4), (5) and (6) of a cotensor functor. Therefore, let $\mathcal{A},\mathcal{B},\mathcal{C} \in \operatorname{obj}(\mathfrak{alg}\mathfrak{Q},\boxtimes)$. We show (4) by proving the commutativity of the simplified diagram

$$\begin{tikzcd}
\widehat{\mathcal{A}\sqcup\mathcal{B}\sqcup\mathcal{C}} \arrow[r, "\mathfrak{E}_{\mathcal{A}\sqcup\mathcal{B},\mathcal{C}}"] \arrow[d, "\mathfrak{E}_{\mathcal{A},\mathcal{B}\sqcup\mathcal{C}}"'] & \widehat{\mathcal{A}\sqcup\mathcal{B}}\otimes\widehat{\mathcal{C}} \arrow[d, "\mathfrak{E}_{\mathcal{A},\mathcal{B}}\otimes\mathrm{id}"] \\
\widehat{\mathcal{A}}\otimes\widehat{\mathcal{B}\sqcup\mathcal{C}} \arrow[r, "\mathrm{id}\otimes\mathfrak{E}_{\mathcal{B},\mathcal{C}}"] & \widehat{\mathcal{A}}\otimes\widehat{\mathcal{B}}\otimes\widehat{\mathcal{C}}
\end{tikzcd}$$

We calculate

$$(\mathfrak{E}_{\mathcal{A},\mathcal{B}}\otimes\mathrm{id})\circ\mathfrak{E}_{\mathcal{A}\sqcup\mathcal{B},\mathcal{C}}(a) = (\mathfrak{E}_{\mathcal{A},\mathcal{B}}\otimes\mathrm{id})(a\otimes p_2) = a\otimes p_2\otimes p_2$$
$$(\mathrm{id}\otimes\mathfrak{E}_{\mathcal{B},\mathcal{C}})\circ\mathfrak{E}_{\mathcal{A},\mathcal{B}\sqcup\mathcal{C}}(a) = (\mathrm{id}\otimes\mathfrak{E}_{\mathcal{B},\mathcal{C}})(a\otimes p_2) = a\otimes p_2\otimes p_2$$

for $a \in \mathcal{A}$ (similarly for $b \in \mathcal{B}$, $c \in \mathcal{C}$ and $p_i \in \mathbb{C}M$, $i \in \{1,2\}$).
Finally, we have to show (5) and (6). We only prove (5), the second property can be shown similarly. For $b \in \mathcal{B}$ and $p_i \in \mathbb{C}M$, $i \in \{1,2\}$ we get

$$(\mathrm{id}\otimes g_0)\circ\mathfrak{E}_{\mathcal{B},\{0\}}(b) = (\mathrm{id}\otimes g_0)(b\otimes p_2) = b = \mathrm{id}_{\widehat{\mathcal{B}}}(b),$$
$$(\mathrm{id}\otimes g_0)\circ\mathfrak{E}_{\mathcal{B},\{0\}}(p_i) = (\mathrm{id}\otimes g_0)(p_i\otimes p_i) = p_i = \mathrm{id}_{\widehat{\mathcal{B}}}(p_i).$$

Obviously, the inclusion $\mathcal{A} \hookrightarrow \widehat{\mathcal{A}}$ is a morphism in the category $\mathfrak{alg}\mathfrak{Q}$.

□

One possible application is the realization of (r,s)-Lévy-processes. We shortly sketch this to see why every part of the above is needed. A detailed version will appear [12].
Fix $r,s \in \mathbb{C}$ and let $\curlywedge$ denote the (r,s)-product. For functionals on a dual semigroup $\mathcal{D}$ this induces a convolution product

$$\varphi_1 \ast \varphi_2 := (\varphi_1 \curlywedge \varphi_2)\circ\Lambda,$$

where Λ is the comultiplication of $\mathcal{D}$.
We use the following theorem, which appears in Franz [7] and is a special case of the general fact that cotensor functors map comonoids to comonoids (see for example [20]).

Theorem 4.2 (Franz, [7]). *Let $\mathcal{F} : (\mathfrak{alg},\sqcup) \to (\mathfrak{alg}_{\mathbb{1}},\otimes)$ be a cotensor functor. If $(\mathcal{D},\Lambda,\delta)$ is a dual semigroup, then $\big(\mathcal{F}(\mathcal{D}),\mathfrak{T}_{\mathcal{D},\mathcal{D}}\circ\mathcal{F}(\Lambda), g_0\circ\mathcal{F}(0)\big)$ is a bialgebra.*

An easy calculation shows that

$$\widehat{\varphi_1 \ast \varphi_2} = \widehat{\varphi_1} \star \widehat{\varphi}_2$$

where $\star$ is the usual convolution for linear functionals on the bialgebra $\widehat{\mathcal{D}}$.
A familiy $(\varphi_t)_{t\geq 0}$ of linear functionals on $\mathcal{D}$ is called *convolution semigroup* if

(i) $\varphi_s * \varphi_t = \varphi_{s+t}$

(ii) $\varphi_0 = \delta$

are fulfilled. It is said to be *weakly continuous*, if

$$\lim_{t\to 0^+} \varphi_t(d) = \delta(d)$$

for all $d \in \mathcal{D}$. In this situation, the linear functionals $\widehat{\varphi}_t$ form a pointwise continuous convolution semigroup on the bialgebra $\widehat{D}$ with respect to the convolution $\star$. By Schürmann's theory of quantum Lévy-processes on bialgebras, one can associate a quantum probability space $(\mathcal{A}, \Phi)$ and a family of algebra homomorphisms $j_{s,t} : \mathcal{D} \to \mathcal{A}$ with $\Phi \circ j_{s,t} = \widehat{\varphi_{t-s}}$ for all $0 \leq s \leq t$. These form a so-called $\otimes$-Lévy-process. The restrictions $k_{s,t} := j_{s,t}|_{\mathcal{D}}$ will form an (r,s)-Lévy-process with marginal distributions $\Phi \circ k_{s,t} = \varphi_{t-s}$ for all $0 \leq s \leq t$. In the last step it is important to have a natural embedding $\mathcal{D} \hookrightarrow \widehat{D}$. That is the reason why a cotensor functor is not enough for our purposes, but one needs a reduction in the sense of Definition 4.4, as in [7].

5 GNS-Modules

In this section we will perform the GNS-construction for the (r,s)-product of two linear functionals. Since the (r,s)-product is not preserving positivity, we have to generalize the usual GNS-construction to not necessarily positive functionals. When comparing with the case of the Boolean product, that is $r = s = 1$, strange things happen to the dimension of the representation space. Since the (r,s)-product of two homomorphisms is not a homomorphism, the dimension can increase, see Example 5.3. It is also possible that the dimensions coincide, as shown in Example 5.4, or even that the dimension is smaller than in the Boolean case, see Example 5.5.

Definition 5.1. A *semi-dual pair* consists of a pair of vector spaces (E, F) and a bilinear form $\langle\cdot,\cdot\rangle : E \times F \to \mathbb{C}$. A semi-dual pair is called a *dual pair* if its bilinear form is *non-degenerate* in the sense that

- $\langle e, f\rangle = 0$ for all $e \in E$ implies $f = 0$ and
- $\langle e, f\rangle = 0$ for all $f \in F$ implies $e = 0$.

In that case the bilinear form is called *dual pairing* of (E, F).

Given a semi-dual pair (E, F) and a subset $M \subset E$, the *orthogonal space* of M is

$$M^\perp := \left\{ f \in F \,\middle|\, \langle e, f\rangle = 0 \ \forall e \in M \right\}$$

and similarly for a subset $N \subset F$

$$N^\perp := \left\{ e \in E \,\middle|\, \langle e, f\rangle = 0 \ \forall f \in N \right\}.$$

The subspaces $F^\perp \subset E$ and $E^\perp \subset F$ are called *degeneracy spaces* of (E, F).

Proposition 5.1. *Let (E, F) be a semi-dual pair. Denote by $[e]$ and $[f]$ the equivalence classes of e and f in $E/F^\perp$ and $F/E^\perp$ respectively. Then*

$$\langle [e], [f] \rangle := \langle e, f \rangle$$

gives a well-defined dual pairing on $(E/F^\perp, F/E^\perp)$.

Proof. Let $e \in E, e' \in F^\perp, f \in F, f' \in E^\perp$. Then

$$\langle e + e', f + f' \rangle = \langle e, f \rangle$$

by bilinearity. This shows well-definedness. To show non-degeneracy assume $\langle [e], [f] \rangle = 0$ for all $f \in F$. Since $\langle [e], [f] \rangle = \langle e, f \rangle$ we get $e \in F^\perp$, hence $[e] = 0$. Analogously, $\langle [e], [f] \rangle = 0$ for all $e \in E$ implies $[f] = 0$. □

Example 5.1. Any complex $m \times n$ matrix B defines a bilinear form

$$\langle x, y \rangle := x^t B y$$

on $\mathbb{C}^m \times \mathbb{C}^n$. We have $F^\perp = \left\{ x \in \mathbb{C}^m \,\middle|\, x^t B y = 0 \ \forall y \in \mathbb{C}^n \right\}$ so $\dim F^\perp = m - \operatorname{rank} B$ and $\dim E/F^\perp = m - \dim F^\perp = \operatorname{rank} B$. Similarly, $\dim F/E^\perp = \operatorname{rank} B$.

Definition 5.2. Let $\mathcal{A}$ be an algebra. A *(semi-)dual pair of $\mathcal{A}$-modules* is a (semi-)dual pair (E, F) such that

- E is a right $\mathcal{A}$-module,
- F is a left $\mathcal{A}$-module,
- $\langle ea, f \rangle = \langle e, af \rangle$ for all $e \in E$, $a \in \mathcal{A}$ and $f \in F$.

If $\mathcal{A}$ has a unit $1_\mathcal{A}$, (E, F) is called *unital* if $e 1_\mathcal{A} = e$ and $1_\mathcal{A} f = f$ for all $e \in E, f \in F$.

Proposition 5.2. *Let $\mathcal{A}$ be an algebra and (E, F) a semi-dual pair of $\mathcal{A}$-modules. If the subspace $U \subset E$ is invariant under the right-action of $\mathcal{A}$ on E, then $U^\perp$ is invariant under the left-action of $\mathcal{A}$ on F.*

Proof. Let $U \subset E$ be invariant, that is $e \in U$ implies $ea \in U$ for all $a \in \mathcal{A}$. For $f \in U^\perp$ and arbitrary $a \in \mathcal{A}$ we get

$$\langle e, af \rangle = \langle ea, f \rangle = 0$$

for all $e \in U$, that is $af \in U^\perp$. □

In particular $F^\perp \subset E$ is invariant. Of course, we can switch the roles of E and F to show $E^\perp \subset F$ is invariant.

Theorem 5.1. *Let (E, F) be a semi-dual pair of $\mathcal{A}$ modules. Then the pair $(E/F^\perp, F/E^\perp)$ is a dual pair of $\mathcal{A}$ modules with actions and dual pairing given by*

$$[e]a = [ea], \qquad a[f] = [af] \quad \textit{and} \quad \langle [e], [f] \rangle = \langle e, f \rangle$$

for all $e \in E, f \in F$ and $a \in \mathcal{A}$.

Proof. The pair $(E/F^\perp, F/E^\perp)$ is a dual pair by Proposition 5.1. The given actions are well-defined by proposition 5.2, . Furthermore, $(E/F^\perp, F/E^\perp)$ is a dual pair of $\mathcal{A}$ modules, since

$$\langle [e]a, [f] \rangle = \langle [ea], [f] \rangle = \langle ea, f \rangle = \langle e, af \rangle = \langle [e], [af] \rangle = \langle [e], a[f] \rangle$$

for all $e \in E, f \in F$ and $a \in \mathcal{A}$. □

Every algebra $\mathcal{A}$ acts on itself from the right and from the left by multiplication. For any linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ the bilinear form $(a,b) \mapsto \varphi(ab) : \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ turns $(\mathcal{A}, \mathcal{A})$ into a semi-dual pair of $\mathcal{A}$-modules. We denote by N_R^φ and N_L^φ the degeneracy spaces. Define $E^\varphi := \mathcal{A}/N_R^\varphi$ and $F^\varphi := \mathcal{A}/N_L^\varphi$. By the preceding theorem (E^φ, F^φ) is a dual pair of $\mathcal{A}$-modules. Furthermore, if $\mathcal{A}$ is unital und $\varphi(1) = 1$, then we define $\Omega^\varphi := [1] \in E^\varphi$ and $\Xi^\varphi := [1] \in F^\varphi$. Then it holds that

$$\langle \Omega^\varphi, a\Xi^\varphi \rangle = \varphi(1a1) = \varphi(a) \tag{8}$$

for all $a \in \mathcal{A}$ and $E^\varphi = \Omega^\varphi \mathcal{A}$, $F^\varphi = \mathcal{A}\Xi^\varphi$.

Definition 5.3. Let $\mathcal{A}$ be an algebra and E a right $\mathcal{A}$-module. A vector $\Omega \in E$ is called *cyclic* if $\Omega\mathcal{A} = E$ and *quasi-cyclic* if $\mathbb{C}\Omega + \Omega\mathcal{A} = E$.

In other words, a vector $\Omega \in E$ is quasi-cyclic if and only if the smallest submodule of E that contains Ω equals E. Cyclicity and quasi-cyclicity for left modules are defined likewise.
For a unital algebra with normalized linear functional (E^φ, F^φ) is a dual pair of $\mathcal{A}$-modules with cyclic vectors $\Omega^\varphi, \Xi^\varphi$ from which we can recover the functional by (8).
Denote by $\widetilde{\mathcal{A}}$ the unitization of $\mathcal{A}$, that is the unital algebra with underlying vector space $\widetilde{\mathcal{A}} = \mathbb{C} \oplus \mathcal{A}$ and product $(\lambda, a)(\mu, b) = (\lambda\mu, \lambda b + a\mu + ab)$. Then a right $\mathcal{A}$-module E can be turned into a right $\widetilde{\mathcal{A}}$-module by setting $e(1,a) := e + ea$. Clearly, a vector $\Omega \in E$ is quasi-cyclic for the $\mathcal{A}$-action if and only if it is cyclic for the corresponding $\widetilde{\mathcal{A}}$-action. For a functional $\varphi : \mathcal{A} \to \mathbb{C}$ define $\widetilde{\varphi} : \widetilde{\mathcal{A}} \to \mathbb{C}$ with $\widetilde{\varphi}(\lambda, a) := \lambda + \varphi(a)$. Using $\widetilde{\mathcal{A}}$ and $\widetilde{\varphi}$ instead of $\mathcal{A}$ and φ we can always use the construction above to find a dual pair of $\mathcal{A}$ modules which allows us to reconstruct φ by (8).

Proposition 5.3. *Let (E,F) be a dual pair of $\mathcal{A}$-modules, $\Omega \in E$, $\Xi \in F$ quasi-cyclic vectors with*

$$\langle \Omega, \Xi \rangle = 1 \quad \textit{and} \quad \langle \Omega, a\Xi \rangle = \varphi(a) \tag{9}$$

for all $a \in \mathcal{A}$. Then there is a unique pair of module isomorphisms $U : E^{\widetilde{\varphi}} \to E$, $T : F^{\widetilde{\varphi}} \to F$ with $U(\Omega^{\widetilde{\varphi}}) = \Omega$ and $T(\Xi^{\widetilde{\varphi}}) = \Xi$. It holds that $\langle Ue, Tf \rangle = \langle e, f \rangle$ for all $e \in E^{\widetilde{\varphi}}, f \in F^{\widetilde{\varphi}}$.

Proof. Since $\Omega^{\widetilde{\varphi}}$ and $\Xi^{\widetilde{\varphi}}$ are quasi-cyclic, U and T are uniquily determined if they exist. We have

$$\left\langle \Omega^{\widetilde{\varphi}} a, b\Xi^{\widetilde{\varphi}} \right\rangle = \varphi(ab) = \langle \Omega a, b\Xi \rangle \quad \text{and} \quad \left\langle \Omega^{\widetilde{\varphi}} a, \Xi^{\widetilde{\varphi}} \right\rangle = \varphi(a) = \langle \Omega a, \Xi \rangle$$

for all $a, b \in \mathcal{A}$. Together this yields $\left\langle \Omega^{\widetilde{\varphi}} a, f \right\rangle = \langle \Omega a, f \rangle$ for all $f \in F$. Provided U and T exist, this also settles the last equality in the proposition. Since $(E^{\widetilde{\varphi}}, F^{\widetilde{\varphi}})$ is a dual pair, the pairing is non-degenerate. So

$$\begin{aligned} \Omega^{\widetilde{\varphi}} a = 0 &\Leftrightarrow \left\langle \Omega^{\widetilde{\varphi}} a, f \right\rangle = 0 \text{ for all } f \in F \\ &\Leftrightarrow \langle \Omega a, f \rangle = 0 \text{ for all } f \in F \\ &\Leftrightarrow \Omega a = 0. \end{aligned}$$

This shows that $U : \lambda \Omega^{\widetilde{\varphi}} + \Omega^{\widetilde{\varphi}} a \mapsto \lambda \Omega + \Omega a$ is well-defined. The existence of T follows analogously. □

Definition 5.4. Let $\mathcal{A}$ be an algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ a linear functional. A dual pair of $\mathcal{A}$-modules (E, F) with quasi-cyclic vectors Ω, Ξ is called *GNS-pair* of $(\mathcal{A}, \varphi)$ if it fulfills (9).

We have seen that a GNS-pair always exists and is unique up to *isometric isomorphism* in the sense of Proposition 5.3. In particular, for $\mathcal{A}$ a unital algebra and $\varphi(1) = 1$ we have $(E^{\varphi}, F^{\varphi}) \cong (E^{\widetilde{\varphi}}, F^{\widetilde{\varphi}})$, since they are both GNS-pairs of $(\mathcal{A}, \varphi)$.

Example 5.2. Let $\mathcal{A}$ be an algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ a homomorphism, that is $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in \mathcal{A}$. Then $(\mathbb{C}, \mathbb{C})$ becomes a dual pair of $\mathcal{A}$-modules with canonical dual pairing $\langle \lambda, \mu \rangle := \lambda\mu$ and left and right actions on $\mathbb{C}$ by

$$\lambda a := \lambda \varphi(a), \quad a\mu := \varphi(a)\mu.$$

The unit $1 \in \mathbb{C}$ is quasi-cyclic for these actions and $(\mathbb{C}, \mathbb{C})$ with $\Omega := 1, \Xi := 1$ is obviously a GNS-pair of $(\mathcal{A}, \varphi)$. Also note that Ω and Ξ are cyclic if and only if $\varphi \neq 0$.

The converse holds as well, that is if the GNS-modules are one-dimensional, then φ is a homomorphism. To show this, let (E, F) with quasi-cyclic vectors $\Omega \in E, \Xi \in F$ be a GNS-pair with $\dim E = \dim F = 1$. So $E = \mathbb{C}\Omega, F = \mathbb{C}\Xi$ and $\langle \lambda\Omega, \mu\Xi = \lambda\mu \rangle$. From $\langle \Omega, a\Xi \rangle = \varphi(a)$ we conclude $a\Xi = \varphi(a)\Xi$. Since $\varphi(ab)\Xi = (ab)\Xi = a(b\Xi) = \varphi(a)\varphi(b)\Xi$ for all $a, b \in \mathcal{A}$, φ is a homomorphism.
Fix $r, s \in \mathbb{C}$ and denote by $\curlywedge$ the (r, s)-product. Let (E_i, F_i) with quasi-cyclic vectors Ω_i, Ξ_i be a GNS-pair of $(\mathcal{A}_i, \varphi_i)$ for $i = 1, 2$. In the following, we will present a way to express the GNS-pair of $(\mathcal{A}_1 \sqcup \mathcal{A}_2, \varphi_1 \curlywedge \varphi_2)$ in terms of the respective GNS-pairs (E_i, F_i). Set

$$E := \mathbb{C}\Omega \oplus \Omega_1\mathcal{A}_1 \oplus \Omega_2\mathcal{A}_2 \quad \text{and} \quad F := \mathbb{C}\Xi \oplus \mathcal{A}_1\Xi_1 \oplus \mathcal{A}_2\Xi_2. \tag{10}$$

and define a semi-dual pairing on (E, F) by

$$\begin{array}{lll} \langle \Omega, \Xi \rangle = 1 & \langle \Omega, b_1\Xi_1 \rangle = \langle b_1 \rangle & \langle \Omega, b_2\Xi_2 \rangle = \langle b_2 \rangle \\ \langle \Omega_1 a_1, \Xi \rangle = \langle a_1 \rangle & \langle \Omega_1 a_1, b_1\Xi_1 \rangle = \langle a_1 b_1 \rangle & \langle \Omega_1 a_1, b_2\Xi_2 \rangle = r\langle a_1 \rangle \langle b_2 \rangle \\ \langle \Omega_2 a_2, \Xi \rangle = \langle a_2 \rangle & \langle \Omega_2 a_2, b_1\Xi_1 \rangle = s\langle a_2 \rangle \langle b_1 \rangle & \langle \Omega_2 a_2, b_2\Xi_2 \rangle = \langle a_2 b_2 \rangle \end{array} \tag{11}$$

where $\langle a\rangle := \varphi_i(a)$ for $a \in \mathcal{A}_i$. Furthermore, set

$$(\lambda\Omega + \Omega_1 a_1 + \Omega_2 a_2)b := \begin{cases} \lambda\Omega_1 b + \Omega_1 a_1 b + s\varphi_2(a_2)\Omega_1 b \text{ for } b \in \mathcal{A}_1 \\ \lambda\Omega_2 b + r\varphi_1(a_1)\Omega_2 b + \Omega_2 a_2 b \text{ for } b \in \mathcal{A}_2 \end{cases} \tag{12}$$

and

$$b(\mu\Xi + c_1\Xi_1 + c_2\Xi_2) := \begin{cases} \mu b\Xi_1 + bc_1\Xi_1 + r\varphi_2(c_2)b\Xi_1 \text{for } b \in \mathcal{A}_1 \\ \mu b\Xi_2 + s\varphi_1(c_1)b\Xi_2 + bc_2\Xi_2 \text{for } b \in \mathcal{A}_2. \end{cases} \tag{13}$$

Theorem 5.2. *The equations* (12),(13) *define actions of $\mathcal{A}_1 \sqcup \mathcal{A}_2$ such that the semi-dual pair (E,F) in* (10) *becomes a semi-dual pair of $\mathcal{A}_1 \sqcup \mathcal{A}_2$-modules. The dual pair $(E/F^\perp, F/E^\perp)$ with the vectors $\Omega + F^\perp, \Xi + E^\perp$ is the GNS-pair of $\varphi_1 \curlywedge \varphi_2$.*

Proof. Straightforward. □

Example 5.3. Suppose $0 \neq \varphi_i : \mathcal{A}_i \to \mathbb{C}$ are homomorphisms. Then by Example 5.2 $E_i = \mathbb{C}\Omega_i$ and $F_i = \mathbb{C}\Xi_i$ for $i = 1,2$. So $E, F \cong \mathbb{C}^3$, and the semi-dual pairing (11) is determined by the matrix

$$B := \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & r \\ 1 & s & 1 \end{pmatrix}$$

in the sense of Example 5.1. We have

$$\operatorname{rank} B = \begin{cases} 1 & \text{for } r,s = 1 \\ 2 & \text{for } r = 1, s \neq 1 \text{ or } r \neq 1, s = 1 \\ 3 & \text{for } r,s \neq 1 \end{cases}$$

so the dimension of the the GNS-pair of $(\mathcal{A}_1 \sqcup \mathcal{A}_2, \varphi_1 \curlywedge \varphi_2)$ depends on r and s.

Can it happen that the semi-dual pairing (11) is degenerate even if $r,s \neq 1$? Before we present two more examples, let us do some general considerations.

Lemma 5.1. Let (E,F) be a GNS-pair of $(\mathcal{A},\varphi)$. Then

$$\Omega\hat{a} = \Omega \Leftrightarrow \varphi(\hat{a}) = 1 \text{ and } \varphi(\hat{a}b) = \varphi(b) \forall b \in \mathcal{A}.$$

Proof. Straightforward. □

Proposition 5.4. *Let $r,s \neq 1$. If Ω_1 and Ω_2 are cyclic, then $E^\perp = \{0\}$.*

Proof. Cyclicity means $\Omega_1\mathcal{A}_1 = E_1$ and $\Omega_2\mathcal{A}_2 = E_2$. Thus, we can rewrite E of (10) as

$$E = \mathbb{C}\Omega \oplus E_1 \oplus E_2.$$

In particular, there exist $\hat{a}_1 \in \mathcal{A}_1, \hat{a}_2 \in \mathcal{A}_2$ with $\Omega_1 = \Omega_1\hat{a}_1, \Omega_2 = \Omega_2\hat{a}_2 \in E$. Let $f = \mu\Xi + b_1\Xi_1 + b_2\Xi_2 \in E^\perp$. Using (11), we get the system of linear equations

$$
\begin{aligned}
\langle \Omega, f\rangle &= \mu + \varphi_1(b_1) + \varphi_2(b_2) = 0\\
\langle \Omega_1, f\rangle &= \mu + \varphi_1(b_1) + r\varphi_2(b_2) = 0\\
\langle \Omega_2, f\rangle &= \mu + s\varphi_1(b_1) + \varphi_2(b_2) = 0
\end{aligned}
$$

which is determinate for $r, s \neq 1$. Hence, $\mu = \varphi_1(b_1) = \varphi_2(b_2) = 0$. Furthermore,

$$\langle \Omega_1 a_1, f\rangle = \varphi_1(a_1 b_1) = \langle \Omega_1 a_1, b_1\Xi_1\rangle = 0$$

for all $a_1 \in \mathcal{A}_1$. Using cyclicity of Ω_1 and non-degeneracy of (E_1, F_1) again, we conclude $b_1\Xi_1 = 0$. In the same way, we get $b_2\Xi_2 = 0$ and finally $f = 0$. ☐

Because of the proposition, we know that in order to get more interesting examples, it is necessary that the quasicyclic vectors for the GNS-pair of at least one of the functionals are not cyclic.

Example 5.4. Let $0 \neq \varphi_1 : \mathcal{A}_1 \to \mathbb{C}$ be a homomorphism and $\varphi_2 : \mathbb{C}_0[x] \to \mathbb{C}$ given by

$$\varphi_2(x^m) := \begin{cases} 1 & \text{for } m = 2\\ 0 & \text{for } m \neq 2. \end{cases}$$

We already know the GNS-pair of φ_1 is $E_1 = \mathbb{C}\Omega_1$ and $F_1 = \mathbb{C}\Xi_1$. To determine the GNS-pair of φ_2 we calculate

$$
\begin{aligned}
\widetilde{N} &= \left\{ p = \sum_{i=0}^{n} \alpha_i x^i \;\middle|\; \widetilde{\varphi_2}(pq) = 0 \quad \forall q \in \mathbb{C}[x] \right\}\\
&= \left\{ p \;\middle|\; \widetilde{\varphi_2}(px^k) = 0 \quad \forall k \in \mathbb{N}_0 \right\}\\
&= \left\{ p = \sum_{i=0}^{n} \alpha_i x^i \;\middle|\; \alpha_0 = \alpha_1 = \alpha_2 = 0 \right\}
\end{aligned}
$$

which yields

$$E_2 = F_2 = {}^{\mathbb{C}[x]}/_{\widetilde{N}} = \operatorname{span}\left\{[1], [x], [x^2]\right\}.$$

Setting $\Omega_2 = \Xi_2 := [1]$ we get

$$\Omega_2\mathbb{C}_0[x] = \operatorname{span}\{\Omega_2 x, \Omega_2 x^2\}, \mathbb{C}_0[x]\Xi_2 = \operatorname{span}\{x\Xi_2, x^2\Xi_2\},$$

and thus,

$$E = \operatorname{span}\{\Omega, \Omega_1, \Omega_2 x, \Omega_2 x^2\}, F = \operatorname{span}\{\Xi, \Xi_1, x\Xi_2, x^2\Xi_2\}$$

with the semidual pairing determined by

$$B = \begin{pmatrix} 1 & 1 & 0 & 1\\ 1 & 1 & 0 & r\\ 0 & 0 & 1 & 1\\ 1 & s & 0 & 0 \end{pmatrix}.$$

We find

$$\operatorname{rank} B = \begin{cases} 3 & \text{for } r = 1 \text{ or } s = 1 \\ 4 & \text{for } r, s \neq 1. \end{cases}$$

So the dimension of the GNS-pair of $\varphi_1 \curlywedge \varphi_2$ can be equal to the one in the Boolean case $r = s = 1$ for other values of r and s.

Example 5.5. As a third example consider $\varphi_1 = \varphi_2 : \mathbb{C}_0[x] \to \mathbb{C}$ with

$$\varphi_i(x^m) := \begin{cases} 1 & \text{for } m = 2 \\ 0 & \text{for } m \neq 2. \end{cases}$$

We already know the GNS-pairs of φ_1 and φ_2. From these we construct

$$E = \operatorname{span}\{\Omega, \Omega_1 x, \Omega_1 x^2, \Omega_2 x, \Omega_2 x^2\}, F = \operatorname{span}\{\Xi, x\Xi_1, x^2\Xi_1, x\Xi_2, x^2\Xi_2\}$$

with the semidual pairing determined by

$$B = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & r \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & s & 0 & 0 \end{pmatrix}.$$

We calculate $\det B = -rs + r + s$ and thus find

$$\operatorname{rank} B = \begin{cases} 4 & \text{for } rs = r + s \\ 3 & \text{otherwise.} \end{cases}$$

This example shows that, surprisingly, the dimension of the GNS-module of $\varphi_1 \curlywedge \varphi_2$ can even be smaller than in the Boolean case $r = s = 1$.

Remark. One may ask if these dimension phenomena can also arise for the normalized universal products when one allows non-positive linear functionals. This it not the case. The same constructions one uses to build the joint GNS-representations of two states from their respective GNS-representations can be applied to build the joint GNS-modules for general linear functionals. Non-degeneracy of the pairings is automatically preserved.

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