THE VICIOUS CIRCLE OF HIGH-SCHOOL GRADUATES' PROOF CONSTRUCTION

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The study aims to get insights into high-school graduates' problems when constructing a proof. Twelve high-school graduates were asked to prove the claim that the sum of an odd natural number and its double is always odd. In the subsequent interviews, it turned out that graduates' minor abilities to cope with algebra and their views on proof hindered them from proving the given claim. Based on graduates' problems during the proving process, I conceptualize the 'vicious circle' of high-school graduates' proof construction. It becomes evident that the graduates nearly had no possibility to prove the given claim. Implications for teaching and learning at school and university are highlighted.

INTRODUCTION

Mathematical proof and proving are internationally known for constituting a challenge for learners of all ages (e.g., Reid and Knipping, 2010). While mathematical reasoning is a prominent part of school-mathematics curriculum worldwide, it has turned out that the learning and teaching of proof remains a problematic issue even at the university level (e.g., Kempen and Biehler, 2019). Based on the aim to get deeper insights into the complex issue of mathematical proof in the transition from school to university, the present study aims to understand high-school graduates' problems when constructing a mathematical proof. Task-based interviews were conducted with 12 high-school graduates, and their problems arising while trying to prove a given claim were analyzed by qualitative content analysis.

BACKGROUND

Different suggestions exist on how mathematical reasoning and proof might be taught in school mathematics. In this sense, a mathematical argumentation (or proof) might be constructed or communicated by making use of symbolic algebraic notation ('symbolic proof'), by making use of generic examples ('generic proof'), or by a narrative ('narrative proof') (see Reid and Knipping (2010, p. 130 ff.)). However, all of these types of proof come with unique challenges, e.g., to cope with algebra, the detection and explication of the generic argument in specific examples, or the verbal description of a mathematical structure and the corresponding generality. The whole proving process consists of several phases (Boero, 1999). In almost all of these phases, checking and investigating concrete examples can positively affect the proving process (Pedemonte & Buchbinder, 2011), while single examples cannot prove a for-all statement. Learning how to construct and communicate proofs is about acquiring respective norms and working in line with them. This is also true regarding signs and notation used in proving processes ("semiotic norms"; Dimmel & Herbst, 2014, p. 393).

FINDINGS FROM THE LITERATURE

Learners' problems with mathematical proofs are well documented (see Reid and Knipping (2010, p. 59 ff.) for an overview). For the case of first-year pre-service teachers, Kempen and Biehler (2019) highlighted students' problems with the concept of proof and their minor acceptance of proofs that are communicated without mathematical symbolic language. These students seemed to hold a semiotic norm about proofs that led them to overemphasize algebraic notation while not focusing on the mathematical arguments.

RESEARCH QUESTIONS

The study aims to get insights into the complex issue of mathematical proof in the transition from school to university. While there are findings on how middle school and university students cope with mathematical proof, I wanted to look at students' proof attempts when graduating from high school. These students went through all the mathematics at school. Respective results give insights into dealing with mathematical proof at the school level and how such ideas might be connected to the university level. Accordingly, the study was driven by the following research questions: (1) What aspects are obstacles in high-school graduates' proving processes when trying to prove a claim from elementary arithmetic? (2) In which way do these obstacles hinder the graduates from accomplishing their proof?

METHOD & DATA ANALYSIS

Task-based interviews were conducted comprising the following phases: (1) posing the question and free work, (2) minimal heuristic suggestions if the learner struggles with the given task, (3) the guided use of heuristic suggestions, and (4) exploratory, metacognitive questions (Goldin, 2000, p. 523). The participants were asked to work on the following task: "Prove that the sum of an odd natural number and its double is always odd". This claim was chosen because it seems accessible to all high-school graduates, it might be solved with content knowledge from middle school, and it allows for different approaches (empirical evidence, generic proofs, symbolic proof, and narrative proof). In the subsequent interview, the researcher gave prompts to make students explain and evaluate their proof productions (e.g., "Tell me, what did you do?"). Twelve high-school graduates (six female, six male; m_{age}=17.92, SD=0.29; six persons taking an advanced mathematics course) participated in the study. All sessions (task completion and subsequent interview) were video recorded, and participants' proof productions were collected. The dialogues were transcribed for further analysis.[1] In the data analysis, I first separated each of the graduates' proof attempts into subparts concerning the type of proof focused: (a) symbolic proof, (b) generic proof, and (c) narrative proof. The qualitative data analysis focused on the aspects of graduates' proof attempts that hindered them from completing a correct mathematical proof (taken from their written work or oral explanations). Corresponding aspects were grouped into categories following the data-driven development of categories (Kuckartz, 2019, p. 184).

RESULTS

A semiotic norm drove (sooner or later) all graduates to try a *symbolic proof*.[2] Here, students faced a problem: no graduate represented an odd natural number in the form $2x + 1$. Accordingly, they could not reach a sum for the odd number and its double in the form $\ldots 2(3x + 1) + 1 = 2m + 1$ (with $= (3x + 1) \in \mathbb{N}$) to explain why the sum is odd. Graduates only calculated $a + 2a = 3a$ and did not

[1] The named research instruments and data sets are described in more detail in Kempen (2022).
[2] As graduate (7) stated: "My math teacher always drilled into me that it is mathematical if you replace numbers with letters and then have a long calculation with letters where the result is at the end".

know how to proceed algebraically. Some graduates explained verbally why $3a$ is an odd number, as the product of two odd numbers is always odd. The existing semiotic norm, however, prevented them from finishing their proof as a mixture of symbolic language and narrative. As the graduates did not complete their symbolic proof, they went on to either look at concrete examples or tried to write down verbally what they found worth mentioning. When looking at concrete examples (with the possibility of constructing a *generic proof*), graduates got insights that could help them to construct a generic proof. However, the status of theorems from elementary arithmetic was unclear to the students. Theorems like “The sum of an odd and an even number is always odd” were sometimes considered mere conjectures that might be true or mere facts or rules that have been learned before. Only some graduates took such theorems for granted in the sense of mathematical knowledge that could be traced back to former definitions of theorems (see Kempen (2022) for more details). But even though graduates were sure about the validity of corresponding statements, they were unsure about the possibility of using them in their present proof attempts[3]. However, on top of all these problems, graduates mentioned the rule “examples are no mathematical proof”. That was why they stopped figuring out a generic proof and went on to a narrative proof. In the case of the *narrative proof*, the former problematic aspects arose again: the status of the theorems from elementary arithmetic was still unclear, and graduates were unsure about what kind of knowledge they might use for the proof. Finally, a semiotic norm drove them back to using algebraic variables.

To sum up, the semiotic norm drove graduates to try a symbolic proof. The lack of algebraic knowledge made it impossible for them to finish their proof in a purely algebraic way. Accordingly, they had to figure out how to work with examples or narratives. There, they faced distinct problems, so they finally returned to try out a symbolic proof again. Taken together, I arrive at what I call the vicious circle of high-school graduates’ proof construction (Figure 1). Only in three cases did graduates find a way out of the ‘vicious circle’ and completed their proof. Their success was always due to the same two aspects. First, they used mathematical knowledge that was absolutely clear to them (like: “If the last digit of a number is odd, then the whole number is odd”). Second, when evaluating their proof, they relied on the verification function of proof. When their proof showed that the claim really holds in every possible case, this aspect was considered higher than the (perceived) semiotic norm about the necessity of using algebra.

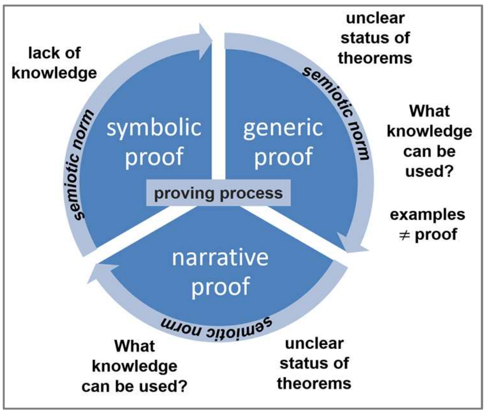

Figure 1: The ‘vicious-circle’ of high-school graduates’ proof construction

[3] As graduate (10) stated: „Still two proofs are missing. One, that the double of an odd number is always even. Exactly, and that the sum of an odd number and an even number is always odd.”

FINAL REMARKS

In this study, I detected the impact of the semiotic norm of using the mathematical symbolic language in high-school graduates' proving processes. Due to their minor abilities to cope with algebra in the context of elementary arithmetic, this semiotic norm hindered them from completing their proving processes. Moreover, graduates' perspective on the role of examples in the proving process seems remarkable. On the one hand, it is correct that single examples cannot serve to prove a for-all statement. However, if this 'rule' is understood superficially, it prevents the students from accomplishing their proof construction. The same phenomenon can be named for the semiotic norm of using the mathematical symbolic language in a proof. The use of algebra has its undeniable advantages, especially concerning the matter of generality. Nevertheless, taken for granted superficially, it prevents successful proving processes. Looking back to school mathematics, it seems reasonable to set up classroom practice that fosters students' abilities to work with algebra and highlights its use for expressing generality. We further experienced the need for a shared set of accepted statements within the given (classroom) community, a more distinct look at the roles of examples in the proving process (especially in the sense of 'generic proofs'), and the value of discussing proofs based on the verification function. For the learning and teaching of proof at the university level, we first face some promising aspects. On the one hand, graduates seemed to hold ideas about the meaning of mathematical proof that appear to be connectable to the concepts of higher mathematics (compare the use of the verification function for judging the proof productions). On the other hand, it has to be stated that these high-school graduates were hardly able to prove the given claim. It becomes clear that these graduates did not have a background in constructing proofs and even struggled with basic proof production. Courses at university should, therefore, provide a meaningful introduction to proof.

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