Transitional Dynamics in R&D-based Models of Endogenous Growth[*]

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Discussion Paper 04/2002

Initial version: July 2002

This version: October 2002

This paper offers a comprehensive study on transitional dynamics within R&D-based models of endogenous growth. There are two main motivations. First, the complete dynamic system for the market solution is derived in general form. Second, using this dynamic system as a unifying framework the adjustment process is analysed. In order to answer the question for the relative importance of transitional dynamics vis-à-vis balanced-growth dynamics, special emphasis is given to the rate of convergence. The investigations show that the models under study can reproduce empirically relevant pattern of development including over- and undershooting as well as growth cycles. The paper demonstrates an alternative route to growth cycles, which does not require complex eigenvalues.

Keywords: Transitional dynamics; R&D-based growth; rate of convergence; dynamic systems; growth cycles

JEL-classification: O0; O3; O31; O33; O41

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[*] For helpful comments and suggestions I thank Lucas Bretschger, Hannes Egli, Rainer Klump, Karl-Josef Koch and Sjak Smulders. Of course, all remaining shortcomings are my own responsibility.

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1. Introduction

Do real-world growth processes mainly represent transitional dynamics or, on the contrary, balanced-growth dynamics? This question is at the heart of the current debate on the relative importance of transitional dynamics vis-à-vis balanced-growth dynamics. The answer to this "relative-importance question" is of major significance for two reasons. First, the empirical implications along the transition path probably differ from those along the balanced growth path. In order to assess the goodness of specific growth models, positive theory accordingly requires a judgement about the relative importance of the two types of dynamics. Moreover, transitional dynamics might enhance our understanding of macroeconomic dynamics in general. At this point there is the chance for a comprehensive theory of macroeconomic dynamics, i.e. an integrated theory of business cycle and growth. Second, the policy implications might be completely different along the transition path from those along the balanced growth path (Jones, 1995a). From a normative point of view it is, therefore, clearly desirable to possess an answer to the question raised above.

Obviously, the answer to the "relative-importance question" under study depends on two components. One concerns the frequency and the severity of macroeconomic shocks, which push the economy from its balanced growth path (or possibly move it even farer away). This is of course a purely empirical issue. The second component concerns the time span which is required to adjust once more closely to the balanced growth path (stability presupposed). The length of the adjustment process is usually described by the half-life time or, equivalently, by the rate of convergence. Of course, a large number of studies have tried to determine the speed of convergence empirically. Since there are, however, substantial problems with this econometric undertaking an independent check on the results is particularly valuable.[1] The paper in hand uses a different approach. A fully calibrated growth model is used to asses the rate of convergence theoretically.

[1] Basically two strands of empirical research can be distinguished. At first, the main flaw of the cross-sectional approaches lies in the fact that an average rate of convergence for very different economies included in the sample is estimated (Temple, 1999). Since the rate of convergence is an endogenous variable this is theoretically unsatisfying. Time-series techniques are more appropriate in this respect. It is well known, however, that the underlying vector-error-correction models are risked to misspecifications of the balanced growth path. In addition, it is unclear how big shocks are dealt with (Ben-David and Papell, 1995). Even more fundamentally, Jones (2002) argues that U.S. economic growth from 1950 to 1993, which is usually identified as balanced growth equilibrium, in fact represents a transition process.

What do we in fact know about transitional dynamics in modern growth models? There are a number of studies dealing with this important issue. At first, the convergence implications of the neoclassical model are quite well understood (Mankiw, Romer and Weil, 1992; Barro and Sala-i-Martin, 1992; King and Rebelo, 1993). In an important contribution, Ortigueira and Santos (1997) investigate the speed of convergence in investment-based endogenous growth models, which focus on the accumulation of human capital. Moreover, Jones (1995a), Eicher and Turnovsky (1999b, 2001) and Perez-Sebastian (2000) examine the quantitative convergence implications of different R&D-based models of growth, the probably most important strand of endogenous growth theory. More specifically, Jones (1995a) analyses the speed of convergence by holding the savings rate and the labour allocation variable between the final-output sector and the R&D sector constant. This procedure simplifies the analysis but hides important transition mechanisms. Eicher and Turnovsky (1999b, 2001) and Perez-Sebastian (2000) investigate the speed at which the economy converges to its balanced growth path. These papers exclusively investigate the social solution. However, since the political system is far from being perfect (even in developed countries) it is clearly indicated to investigate the decentral economy as far as positive theory is concerned.

The paper in hand enhances the literature on transitional dynamics in endogenous growth models by focusing on the market solution.[2] It highlights both the quantitative and the qualitative convergence implications. The quantitative convergence implications, as expressed by the rate of convergence, are important in assessing the general meaning of transitional dynamics. The qualitative convergence implications concern aspects of monotonic versus non-monotonic adjustments and represent empirically relevant pattern of economic development. The class of models under study comprises R&D-based endogenous growth models of the increasing-variety type. Of course, the Romer (1990a) model appears as the natural starting point for this line of research. However, this model is clearly overrestrictive since it imposes a double knife-edge restriction on the R&D technology and further constrains population to be stationary. More importantly, the model bears the scale-effect implication, which can be easily falsified empirically (Jones, 1995a, 1995b). Jones (1995a) generalises the Romer model in that the above mentioned restrictions are relaxed and the scale-effect implication is removed. Yet, even the Jones model is unnecessarily restrictive since it constrains the elasticity of technology to equal the elasticity of labour in the production of output. The basic non-scale R&D-based growth model (Eicher and Turnovsky, 1999b) further

[2] As far as comparative aspects are concerned, the social solution is investigated as well.

relaxes this constraint. This model is therefore used as the general workhorse to analyse transitional dynamics within R&D-based endogenous growth models. Moreover, it is clearly desirable to know how the results are affected by further generalising the basic set-up. The most obvious extension concerns the possibility that capital goods are considered to be productive in R&D as well. To answer this question, a generalised non-scale R&D-based growth model is additionally employed.

The analysis proceeds in several steps. First, the complete dynamic system governing the evolution of the market solution for a wide class of R&D-based growth models is derived in general form. Second, this dynamic system is subsequently reformulated in scale-adjusted variables to receive a stationary system. Third, the stationary solution of the scale-adjusted system is determined. The eigenvalues of the dynamic system are calculated. At this stage the asymptotic rate of convergence can be determined and the stability properties can be checked numerically. Extensive sensitivity analyses is conducted to asses whether the results are robust with respect to parameter changes. Fourth, the differential equation system is solved by backward integration. The characteristic properties of the adjustment processes are illustrated by discussing the resulting simulation results.

The paper is organised as follows. In section 2 a general R&D-based model of the increasing variety type is developed. The quantitative and the qualitative convergence implications are investigated in section 3. Finally, section 4 offers a summary and conclusion.

2. A general R&D-based model

2.1. The basic structure

The market equilibrium for the class of R&D-based growth models of the increasing-variety type is developed and the differential equation system governing the dynamics of the decentral solution is derived.[3] The model is general in two respects. First, general formulations are used as far as possible and restrictions on the formal structure of the model are only introduced provided that these become necessary. Second, each factor of production (labour, capital and technological knowledge) is allowed to be productive in each sector (final output, intermediate goods and R&D).

At first, it is helpful to sketch the structure of the economy under consideration. On the production side there are three sectors. First, the final output sector produces a homogenous

[3] This section concisely develops a general R&D-based growth model; for details see Steger (2002).

good that can be used for consumption or investment purposes. Second, the intermediate goods sector produces differentiated intermediate goods that serve as an input in the production of final output. Third, the R&D sector searches for ideas (designs), which are the technical prerequisite to produce new intermediate goods. Households choose their level of consumption and inelastically supply one unit of labour at every point in time.

Let us now turn to the formal structure of the model. The state variables are the stock of physical capital (K) and the number of designs (A). The model comprises three choice variables, namely the level of consumption (C), the share of labour (θ) and the share of capital (ϕ) devoted to the production of final output. Finally, since we have three distinct goods, there are three prices. Final output serves as the numeraire, its price is set equal to unity. The price of the typical intermediate good is denoted by p and the price of a typical design by v respectively. The order of the dynamic system can be reduced by eliminating the price of intermediate goods (p).

2.2. Firms

Final output sector

The final output sector is assumed competitive and produces a homogenous good that can be used universally for consumption or investment purposes. The original production function may be expressed as $Y = \overline{\overline{F}}\left[\theta L, \phi(i)x(i), A\right]$, where Y denotes final output, L the stock of labour and $x(i)$ with i real valued and $i \in [0, A]$ denotes the number of differentiated intermediate goods of type i. The parameter A indicates the number of differentiated capital goods available at every point in time. A characteristic feature of this class of models is that this number is an endogenous variable; the law of motion of A is described below. The allocation variables θ and $\phi(i)$ $[0 \leq \theta, \phi(i) \leq 1]$ represent the shares of labour and intermediate goods allocated to final output production respectively. Moreover, the final output technology satisfies $\overline{\overline{F}}_{\theta L}(.) > 0$, $\overline{\overline{F}}_{\phi(i)x(i)}(.) > 0$, $\overline{\overline{F}}_{A}(.) > 0$, where $\overline{\overline{F}}_{\theta L}(.)$ is a short-hand symbol for the partial derivative, i.e. $\overline{\overline{F}}_{\theta L}(.) := \dfrac{\partial \overline{\overline{F}}(.)}{\partial \theta L}$. Since it is further assumed that the differentiated intermediate goods enter the production function symmetrically (which is a simplifying assumption), the index i can be ignored and the original production function may be expressed as $Y = \overline{F}\left(\theta L, \phi x, A\right)$. In addition, the production function is required to satisfy three further restrictions: (1) Constant returns to scale in the private inputs (L and x) in order to enable a competitive equilibrium in the final-output sector. (2) The differentiated intermediate goods substitute imperfectly for each other, i.e. the elasticity of substitution is finite. (3) The number of intermediate inputs causes total factor productivity to rise. This feature formalises the basic idea of an increasing productivity due to the division of labour (Ethier, 1982).

The production technology of the intermediate good sector (to be described in next section) implies that aggregate capital can be sensibly defined as $K := q\,A\,x$, where q represents a constant technology parameter. By substituting $x = K/(q\,A)$, the original production function, $Y = \bar{F}\left(\theta L, \phi x, A\right)$, can then be transformed to read $Y = F\left(\theta L, \phi K, A\right)$. The reason for the distinction between the original and the transformed production function lies in the fact that the former underlies basic relations, which describe the market equilibrium (e.g. the demand function for x). The latter formulation must be used to describe the dynamics of the aggregate capital stock given by $\dot{K} = F\left(\theta L, \phi K, A\right) - \delta_K K - C$, where $\delta_K \geq 0$ denotes the constant rate of capital depreciation and C total consumption.

Intermediate goods sector

This sector is composed of an infinite number of firms ordered on the interval $[0, A]$ who manufacture differentiated intermediate goods. Each producer must at first invest in blueprints (designs) as the technical prerequisite of production. As a result of an effective patent protection, the owner of a blueprint is the only producer of the respective intermediate good. The representative intermediate goods producer can convert q units of final output into one unit of the differentiated producer good; of course $q > 0$. Operating profits of the representative intermediate goods producer may then be expressed as $\pi(x) = \left[p(x) - q\,r\right]x$. The gross interest rate is denoted by r, i.e. $r = r_n + \delta_K$ with r_n representing the net interest rate.

The typical intermediate good producer faces two demand schedules. One stems from final output producers, while the other originates from R&D firms. It is assumed that the elasticities of substitution among the intermediate goods are constant for both the final output as well as the R&D sector. Furthermore, since there is a large number of firms in both sectors, the elasticities of substitution equal the respective price elasticities of demand denoted by ε_1

(final output) and ε_2 (R&D). To simplify matters we assume $\varepsilon_1 = \varepsilon_2 = \varepsilon$, so that the intermediate goods producer has no incentives to differentiate prices. With constant marginal costs ($q\,r$) and a price elasticity given by ε, the solution to the underlying monopoly pricing problem implies a supply price of $p_S = \frac{\varepsilon}{\varepsilon - 1} q\,r$. At this stage it becomes obvious that $1 < \varepsilon < \infty$ is necessary to guarantee positive profits.

To derive the profit and the interest rate in terms of the state variables, we need to specify the demand schedules for intermediate goods. The final output sector is assumed competitive and, hence, the typical producer is willing to pay the marginal products for his/ her inputs. The inverse (conditional) demand functions for intermediate goods originating from the final-output sector are given by $p_D(i) = \overline{\overline{F}}_{\phi(i)x(i)}\left[\theta L, \phi(i)x(i), A\right]$ for all i. Since all $x(i)$ enter the production function symmetrically, we can drop the index i and write the demand function as $p_D = \overline{\overline{F}}_{\phi x}\left[\theta L, \phi x, A\right]$. Moreover, since we wish to express the dynamics of the model in terms of aggregate rather than disaggregate capital, we substitute $x = K/(q\,A)$ into $\overline{\overline{F}}_{\phi x}(\theta L, \phi x, A)$ to get $G(\theta L, \phi K, A)$. This function shows the marginal product of one specific variety of the intermediate good in the production of final output in terms of K.

From $\pi(x) = \left[p(x) - q\,r\right]x$, $p_D = G(\theta L, \phi K, A)$, $p_S = \frac{\varepsilon}{\varepsilon - 1} q\,r$, $p(x) = p_D = p_S$ and $x = K/(q\,A)$ operating profits can be expressed to read $\pi = \frac{G(\theta L, \phi K, A)K}{\varepsilon\, q\, A}$. Moreover, from equilibrium in the intermediate goods market ($p_D = p_S$), we have $G(\theta L, \phi K, A) = \frac{\varepsilon}{\varepsilon - 1} q\,r$ and hence the interest rate may be expressed as $r = \frac{\varepsilon - 1}{\varepsilon} \frac{G(\theta L, \phi K, A)}{q}$.

R&D sector

There is a large number of R&D firms who search for new ideas (designs). The R&D technology is of the following shape $\dot{A} = \overline{\overline{\overline{J}}}\left\{A, (1-\theta)L, \overline{(1-\theta)L}, \left[1-\phi(i)\right]x(i), \overline{\left[1-\phi(i)\right]x(i)}\right\}$. This general formulation deserves a thorough explanation. At first, it should be noted that this production function generalises the usual R&D technology in that intermediate goods [$x(i)$] are considered to be productive in R&D as well. Second, it is assumed that $\overline{\overline{\overline{J}}}_A(.) > 0$ which captures two distinct effects. On the one hand, A indicates the net effect of (intertemporal) knowledge spill-overs and "fishing out" effects (Jones and Williams, 2000). On the other hand, in case of capital being productive in R&D, A additionally reflects the specialisation effect due to the use of differentiated producer goods. Third, $\overline{\overline{\overline{J}}}_{(1-\theta)L}(.) > 0$ and $\overline{\overline{\overline{J}}}_{[1-\phi(i)]x(i)}(.) > 0$ denote the private marginal product of labour and differentiated capital goods respectively. It is assumed that there are constant returns to scale at the level of the individual firm. Fourth, following Jones (1995a) and Jones and Williams (2000) we allow for negative externalities associated with the economywide averages of the private resources. The economywide averages of private resources are denoted by $\overline{(1-\theta)L}$ and $\overline{\left[1-\phi(i)\right]x(i)}$. The negative externalities associated with these averages are indicated by $\overline{\overline{\overline{J}}}_{\overline{(1-\theta)L}}(.) \leq 0$ and $\overline{\overline{\overline{J}}}_{\overline{[1-\phi(i)]x(i)}}(.) \leq 0$.[4] These capture (intratemporal) duplication externalities which may be either accidental or intentional (like in the case of R&D races). As before, we assume that the $x(i)$ enter the production function symmetrically. Hence, we can drop the index i and simplify the preceding function by writing $\dot{A} = \overline{\overline{J}}\left[A, (1-\theta)L, \overline{(1-\theta)L}, (1-\phi)x, \overline{(1-\phi)x}\right]$. Moreover, using $x = K/(q\,A)$ allows us to transform this function to read $\dot{A} = \overline{J}\left[A, (1-\theta)L, \overline{(1-\theta)L}, (1-\phi)K, \overline{(1-\phi)K}\right]$. Since in equilibrium $\left(1-\theta\right)L = \overline{\left(1-\theta\right)L}$ and $(1-\phi)K = \overline{(1-\phi)K}$ we may express the preceding function as $\dot{A} = J\left[A, (1-\theta)L, (1-\phi)K\right]$.

An example should clarify the issue. The specific R&D function may take the form $\overline{\overline{\overline{J}}}(.) = \alpha_J\, A^{\eta_{SA}} \left[\left(1-\theta\right)L\right]^{\eta_L^p} \left[\overline{\left(1-\theta\right)L}\right]^{\eta_L^e} \int_0^A \left\{\left[1-\phi(i)\right]x(i)\right\}^{\eta_K^p} \left\{\overline{\left[1-\phi(i)\right]x(i)}\right\}^{\eta_K^e} di$, where η_L^p measures the private effect of labour and η_L^e the external effect associated with the economywide average of labour in R&D. Similarly, η_K^p measures the private effect of capital and η_K^e the external effect associated with the economywide average of capital in R&D. Noting the general symmetry among the $x(i)$ leads to

[4] To clarify notation: $\overline{\overline{\overline{J}}}_{\overline{(1-\theta)L}}(.)$ means $\overline{\overline{\overline{J}}}_{\overline{(1-\theta)L}}(.) := \dfrac{\partial \overline{\overline{\overline{J}}}(.)}{\partial \overline{(1-\theta)L}}$.

$\overline{\overline{J}}(.) = \alpha_J\, A^{\eta_{SA}} \left[\left(1-\theta\right)L\right]^{\eta_L^p} \left[\overline{\left(1-\theta\right)L}\right]^{\eta_L^e} A\left[\left(1-\phi\right)x\right]^{\eta_K^p} \left[\overline{\left(1-\phi\right)x}\right]^{\eta_K^e}$. Considering $x = K/(q\,A)$ allows us to write $\overline{J}(.) = \alpha_J A^{\eta_A} \left[\left(1-\theta\right)L\right]^{\eta_L^p} \left[\overline{\left(1-\theta\right)L}\right]^{\eta_L^e} \left[\left(1-\phi\right)K\right]^{\eta_K^p} \left[\overline{\left(1-\phi\right)K}\right]^{\eta_K^e} q^{-\eta_K}$ with $\eta_A := \eta_{SA} + 1 - \eta_K$. The exponent η_A captures the net effect of the positive spill-over effect and the fishing out effect (η_{SA}) as well as the specialisation effect ($1-\eta_K$). Since in equilibrium $\left(1-\theta\right)L = \overline{\left(1-\theta\right)L}$ and $(1-\phi)K = \overline{(1-\phi)K}$ we may express the preceding function as $J(.) = \alpha_J A^{\eta_A} \left[\left(1-\theta\right)L\right]^{\eta_L} \left[\left(1-\phi\right)K\right]^{\eta_K} q^{-\eta_K}$ with $\eta_L := \eta_L^p + \eta_L^e$ and $\eta_K := \eta_K^p + \eta_K^e$.

The typical R&D firm sets the price of one design to extract the present value of the infinite profit stream accruing at first to the typical intermediate good producer. Hence, this price is given by $v(t) = \int_t^\infty \pi(\tau)\, e^{-R(t)}\, d\tau$ with $R(t) := \int_t^\tau r_n(u)\, du$. The price of one design equally shows the value of the representative intermediate goods firm. Here we have the second market distortion since only private returns are counted and positive spill-over effects are ignored. Differentiating the preceding integral equation with respect to time gives $\dot{v} = r_n\, v - \pi$. This equation can be interpreted as the no-arbitrage condition for the two financial assets existing in this model. The reward of a consumption loan of size v amounts to $r_n\, v$, while the reward of an equity (issued by intermediate goods producers) of equal size is given by $\dot{v} + \pi$. Inserting the expressions for π and r derived above, one obtains the differential equation in v as $\dot{v} = \left[\frac{\varepsilon-1}{\varepsilon}\frac{G\left(\theta L, \phi K, A\right)}{q} - \delta_K\right] v - \frac{G\left(\theta L, \phi K, A\right)K}{\varepsilon\, q\, A}$.

Let us now turn to the factor allocation conditions. Profit-maximising firms reward the factors of production according to their (private) marginal product. Moreover, in equilibrium wages are equalised across the two sectors so that $w = F_{\theta L}(.) = v\, J_{(1-\theta)L}(.)$. This intersectoral labour allocation condition may be expressed as $\theta = \theta(A, K, L, v)$. As for the differentiated capital goods, we have $p_D = \overline{\overline{F}}_{\phi x}(.) = v\, \overline{\overline{J}}_{(1-\phi)x}(.)$; notice that $\overline{\overline{J}}_{(1-\phi)x}(.)$ requires to differentiate $\overline{\overline{J}}(.)$ with respect to $\left[1-\phi(i)\right]x(i)$ and then drop the index i. Substituting $x = K/(q\,A)$ into the preceding equation gives the allocation condition for intermediate goods in terms of (aggregate) capital as $\phi = \phi(A, K, L, v)$.

A comment on generality

It is fairly obvious that only a limited number of specific production functions fit into the general framework stated above. In order to clarify this aspect further, the following consideration may be instructive. The functions meeting the preceding requirements may be expressed as $Y = B[\theta_1 L_1, \theta_2 L_2] Z[\phi(i) x(i)]$.[5] The subfunction $Z[\phi(i) x(i)]$ must be CES in the $\phi(i) x(i)$ with an elasticity of substitution (ε) satisfying $1 < \varepsilon < \infty$. It follows that the Cobb-Douglas case is not admissible. The subfunction $B[\theta_1 L_1, \theta_2 L_2]$ could equally be of the CES type or, more specifically, Cobb-Douglas in $\theta_1 L_1$ and $\theta_2 L_2$, where L_1 and L_2 could represent different types of labour.[6]

2.3. Households

The representative household is assumed to inelastically supply one unit of labour during every period of time and to maximise his/ her intertemporal utility. The instantaneous utility function is of the constant-intertemporal-elasticity-of-substitution type (CIES); a specific formulation is used to reduce notational effort. The dynamic optimisation problem reads as follows (see also Jones, 1995a, p. 782).

$$\max_{\{C/L\}} \int_0^\infty \frac{(C/L)^{1-\gamma} - 1}{1-\gamma} e^{-\rho t} \, dt$$

$$s.t.\ \ \dot{K} = r_n K + wL + A\pi - v\dot{A} - C;\ K(0) > 0, \tag{1}$$

where $\rho > 0$ denotes the constant time preference rate, $\gamma > 0$ a constant preference parameter and w the wage rate respectively. From the first-order conditions we get the Keynes-Ramsey rule describing the optimal consumption profile.[7]

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[5] The following remarks apply to the production functions in original form.

[6] In an extension to his original approach, Romer (1990b, p. 347) uses the production technology $Y = g(H, L) \int_0^A x(i)^{\sigma_K} di$, where $g(H, L)$ denotes a CES function. The same considerations apply to the R&D technology.

[7] It is assumed that the sufficiency condition is equally satisfied. In addition, the transversality condition demands for the following inequality constraint to be met $-\rho + \lim_{t\to\infty} \hat{\lambda} + \lim_{t\to\infty} \hat{K} < 0$, where λ denotes the current-value shadow price of capital.

$$\dot{C} = \frac{C}{\gamma}\left[r_n - \rho - (1-\gamma)n\right] \tag{2}$$

2.4. The dynamic system

The preceding discussion can be summarised by the following set of equations. The system shown below governs the dynamics of the market solution for a broad class of R&D-based endogenous growth models of the increasing-variety type. It should be noted that this is a differential-algebraic system of equations. The factor allocation conditions are given in integrated form. Moreover, the labour and capital allocation variables in most cases represent implicit equations. Since these depend, inter alia, on v, the dynamics of the equity price need to be taken into consideration. The price of intermediate goods (p) has been eliminated.

$$\dot{K} = F\left[A,\, \theta L,\, \phi K\right] - \delta_K K - C \tag{3}$$

$$\dot{A} = J\left[A,\, (1-\theta)L,\, (1-\phi)K\right] \tag{4}$$

$$\dot{C} = \frac{C}{\gamma}\left[\frac{\varepsilon-1}{\varepsilon}\frac{G\left(\theta L, \phi K, A\right)}{q} - \delta_K - \rho - (1-\gamma)n\right] \tag{5}$$

$$\dot{v} = \left[\frac{\varepsilon-1}{\varepsilon}\frac{G\left(\theta L, \phi K, A\right)}{q} - \delta_K\right]v - \frac{G\left(\theta L, \phi K, A\right)K}{\varepsilon\, q\, A} \tag{6}$$

$$\theta = \theta(A,\, K, L, v) \tag{7}$$

$$\phi = \phi(A, K, L, v) \tag{8}$$

The size of population (L) is assumed to grow at exponential rate, i.e. $\dot{L} = nL$. The function $G(\theta L, \phi K, A)$ shows the marginal product of one specific variety of intermediate goods in the production of final output in terms of K; formally this function results from the substitution of $x = K/(q\,A)$ into $\overline{\overline{F}}_{\phi x}(\theta L, \phi x, A)$.

Specific models which are included in this general formulation comprise the first-generation of R&D-based models like the original Romer (1990a) model, the non-scale models of Jones (1995a) and Eicher and Turnovsky (1999b, 2001). Further examples are the

CES-CES technology used in Romer (1990b) as well as models with complementarities among intermediate goods (Benhabib and Xie, 1994).

2.5. The balanced growth path

As usual a balanced growth path is defined by constant, though possibly different, growth rates of the endogenous variables. This definition implies that the allocation variables (θ and ϕ) must be constant along the balanced growth path. At this point we can apply the procedure used in Eicher and Turnovsky (1999a). In accordance with the stylised facts, we use the auxiliary assumption stating that $\hat{Y} = \hat{K}$ along the balanced growth path (Romer, 1989). From $\hat{K} = Y/K - \delta_K - C/K$ it then follows that balanced-growth further requires $\hat{K} = \hat{C}$. The balanced growth rates of K and A can be derived from $\frac{d}{dt}\frac{F(.)}{K} = 0$ and $\frac{d}{dt}\frac{J(.)}{A} = 0$ by noting that the allocation variables are constant. Carrying out the preceding instructions yields.

$$(1-\sigma_K)\,\hat{K} - \sigma_A\,\hat{A} = \sigma_L\,\hat{L} \tag{9}$$

$$(1-\eta_A)\,\hat{A} - \eta_K\,\hat{K} = \eta_L\,\hat{L} \tag{10}$$

The elasticities of production σ_x and η_x are defined by $\sigma_x := \frac{F_x(.)\,x}{F(.)}$ and $\eta_x := \frac{J_x(.)\,x}{J(.)}$ for $x = A, L, K$. These are exogenous constants in the Cobb-Douglas case and a function of the input vector in the more general CES case. Provided that $\hat{L} = n > 0$ equations (9) and (10) uniquely determine $\hat{K}$ and $\hat{A}$ given as follows.

$$\hat{K} = \beta_K\, n \text{ with } \beta_K = \frac{\sigma_L\,(1-\eta_A) + \eta_L\,\sigma_A}{(1-\eta_A)(1-\sigma_K) - \eta_K\,\sigma_A} \tag{11}$$

$$\hat{A} = \beta_A\, n \text{ with } \beta_A = \frac{\eta_L\,(1-\sigma_K) + \eta_K\,\sigma_L}{(1-\eta_A)(1-\sigma_K) - \eta_K\,\sigma_A} \tag{12}$$

Eicher and Turnovsky (1999a, section 2.1) derive the conditions for positive and balanced growth applying to the social solution of a general R&D-based growth model. Since the underlying production functions and the resulting balanced growth rates are structurally identical for the market and the social solution, these results can be applied here as well. According to their proposition 1, $(1-\eta_A)(1-\sigma_K)-\eta_K\,\sigma_A>0$ and $\sigma_K<1$ is necessary and sufficient for positive growth. In proposition 2 they summarise three conditions each of which guarantees balanced growth; these are subsequently restated. The production functions in both sectors must be either: (1) constant returns to scale; (2) of the Cobb-Douglas type or (3) homogenously separable in the exogenously and endogenously growing factors.

Three points are especially worth being noticed at this stage. First, balanced growth is characterised by non-scale growth, i.e. the scale of the economy does not influence the pace of growth. Second, the model shows even growth ($\hat{K}=\hat{A}$) in the first and the third case and uneven growth ($\hat{K}\neq\hat{A}$) in the second case. Third, the balanced growth rates of the market and the social solution coincide provided that $\hat{L}=n>0$ and both production functions are of the Cobb-Douglas type. This proposition follows from the fact that, first, the production functions are identical from the perspective of the individual actors and the social planner and, second, in the Cobb-Douglas case the elasticities of production (σ_x and η_x) are exogenous constants.

2.6. The dynamic system in scale-adjusted variables

We now perform an adjustment of scale to receive a dynamic system which possesses a stationary solution and to obtain a convenient expression for the balanced growth path. In order to illustrate this procedure, consider a variable $X(t)$ which grows in the long run at constant rate g, i.e. $\lim_{t\to\infty}\dfrac{\dot{X}(t)}{X(t)}=g$. By defining a new variable $x(t)$, we can then perform an adjustment of scale yielding the scale-adjusted variable $x(t):=\dfrac{X(t)}{e^{g\,t}}$. By construction, $x(t)$ converges to its stationary value denoted by $\tilde{x}$ as time approaches infinity, i.e. $\lim_{t\to\infty}x(t)=\tilde{x}$. Using the definition above, the growth path of $X(t)$ is given by $X(t)=x(t)\,e^{g\,t}$ while the balanced growth path reads $\tilde{X}(t)=\tilde{x}\,e^{g\,t}$.

With the balanced-growth rates shown in (11) and (12), the appropriate scale adjustments are given by $y:=Y/L^{\beta_K}$, $k:=K/L^{\beta_K}$, $c:=C/L^{\beta_K}$, $a:=A/L^{\beta_A}$, $j:=J/L^{\beta_A}$ and

$v_a := v / L^{\beta_K - \beta_A}$.[8] For the class of models considered in this paper it holds true that $\varepsilon = (1-\sigma_K)^{-1}$ and hence $(\varepsilon - 1)/\varepsilon = \sigma_K$. Moreover, in this case we can make use of $G(\theta L, \phi K, A) = \dfrac{\sigma_K F(.)\, q}{\phi K}$. The dynamic system in scale-adjusted variables may consequently be expressed as follows.

$$\dot{k} = y - c - \delta_K k - \beta_K n k \tag{13}$$

$$\dot{a} = j - \beta_A n a \tag{14}$$

$$\dot{c} = \frac{c}{\gamma}\left[\frac{\sigma_K{}^2 y}{\phi k} - \delta_K - \rho - (1-\gamma) n\right] - \beta_K n c \tag{15}$$

$$\dot{v}_a = v_a\left[\frac{\sigma_K{}^2 y}{\phi k} - \delta_K - (\beta_K - \beta_A) n\right] - \frac{(1-\sigma_K)\sigma_K y}{\phi a} \tag{16}$$

$$\sigma_L \frac{y}{\theta} = v_a \frac{\eta_L^p j}{1-\theta} \tag{17}$$

$$\sigma_K \frac{y}{\phi} = v_a \frac{\eta_K^p j}{1-\phi} \tag{18}$$

Three points should be observed at this stage. First, since the adjustment of scale makes the output functions in scale-adjusted variables (y and j) independent of L, this procedure reduces the dimension of the dynamic system. Second, from the definition $\sigma_x := \dfrac{f_x(.)\, x}{f(.)}$ we can express the derivative of the production function with respect to x as $f_x(.) = \dfrac{\sigma_x f(.)}{x}$. Third, η_L^p and η_K^p denote the elasticity of private labour and capital respectively.[9]

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[8] The scale-adjusted price ($v_a := v / L^{\beta_K - \beta_A}$) results from the following consideration. From (6) together with $G(.) = \dfrac{\sigma_K Y q}{\phi K}$, the growth rate of v may be expressed as $\hat{v} = \dfrac{\varepsilon - 1}{\varepsilon}\dfrac{\sigma_K Y q}{\phi K} - \delta_K - \dfrac{\sigma_K Y}{\varepsilon A \phi K v}$. Along the balanced growth path, the first term on the RHS is constant. Hence, v must grow at a rate equal to $\hat{v} = \hat{Y} - \hat{A} = (\beta_K - \beta_A) n$ along the balanced growth path.

[9] Remember that we allow for negative externalities of the private resources employed in R&D. The elasticity of labour in R&D is $\eta_L := \eta_L^p + \eta_L^e$, where η_L^p measures the private effect of labour in R&D and $\eta_L^e \le 0$ measures

3. Transitional dynamics

3.1. Motivation and a basic concept

What are the causes of transitional dynamics from an economic perspective? There are two sources of shocks that give rise to adjustment dynamics. First, suppose that all relevant economic parameters are fixed and, for whatever reason, the economy starts with a combination of state variables which does not coincide with the stationary solution. A reasonable example may be a war or a natural catastrophe that destroys physical, human and knowledge capital. Second, one can argue that the economy finds itself in its long-run equilibrium initially. A sudden change in technology or preference parameters may occur subsequently. As a result, the initial state deviates from the new long-run equilibrium. Provided that the system is stable, the economy converges towards the new equilibrium.

Since the speed of convergence plays a crucial role in this paper this concept is described concisely. The speed at which some variable converges to its equilibrium value is measured by the rate of convergence. Obviously we are dealing with convergence in the sense of the conditional β-convergence hypothesis (Sala-i-Martin, 1996). The rate at which some variable $x(t)$ converges to its balanced growth path $\tilde{x}(t)$ is measured by $\psi_x(t) := -\frac{\dot{x}(t) - \dot{\tilde{x}}(t)}{x(t) - \tilde{x}(t)}$. If the variable under study converges (diverges), then $\psi_x(t) > 0$ [$\psi_x(t) < 0$].[10] The (instantaneous) rate of convergence may be variable along the transition. In the limit, however, the rate of convergence is constant.

3.2. The basic non-scale model of R&D-based growth

3.2.1. The model

Consider now the basic non-scale R&D-based model. The model is characterised by Cobb-Douglas technologies in both sectors of production. There are no further restrictions which are specific to this model. Of course, the set of general restrictions which are necessary

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the external effect associated with the economywide average of labour in R&D. Analogous explanations apply to capital in R&D ($\eta_K := \eta_K^p + \eta_K^e$).

[10] This proposition holds true irrespective of the fact whether $x(t)$ converges from below or from above. It should also be noted that this definition does not require the balanced-growth equilibrium to be stationary.

for the existence of the market solution, internal consistency as well as positive and balanced-growth apply. The production side of the economy in terms of aggregate capital is given as follows.[11]

$$Y = F(.) = \alpha_F A^{\sigma_A} (\theta L)^{\sigma_L} K^{\sigma_K} \quad \text{with} \quad \sigma_A, \sigma_L, \sigma_K > 0\,;\; \sigma_L + \sigma_K = 1 \tag{19}$$

$$\dot{A} = J(.) = \alpha_J A^{\eta_A} [(1-\theta) L]^{\eta_L} \quad \text{with} \quad \eta_A, \eta_L > 0\,;\; \eta_L \le 1 \;(\eta_L^p = 1,\; \eta_L^e \le 0) \tag{20}$$

$$\dot{L} = n L \tag{21}$$

The balanced-growth rates turn out to read $\hat{K} = \hat{Y} = \hat{C} = \dfrac{\left[(1-\eta_A)\sigma_L + \eta_L \sigma_A\right] n}{(1-\eta_A)(1-\sigma_K)} = \beta_K\, n$ and $\hat{A} = \dfrac{\eta_L\, n}{1-\eta_A} = \beta_A\, n$. Necessary and sufficient conditions for positive per capita growth are $(1-\eta_A)(1-\sigma_K) > 0$ and $\sigma_K < 1$.[12]

Let us now turn to the dynamic system in scale-adjusted variables. From (19), (20) together with $y := Y / L^{\beta_K}$, $k := K / L^{\beta_K}$, $c := C / L^{\beta_K}$, $a := A / L^{\beta_A}$, $j := J / L^{\beta_A}$, $v_a := v / L^{\beta_K - \beta_A}$ and considering $\beta_K = \dfrac{(1-\eta_A)\sigma_L + \eta_L \sigma_A}{(1-\eta_A)(1-\sigma_K)}$ and $\beta_A = \dfrac{\eta_L\, n}{1-\eta_A}$ we can derive the production functions in scale-adjusted variables to read $y = \alpha_F a^{\sigma_A} \theta^{\sigma_L} k^{\sigma_K}$ and $j = \alpha_J a^{\eta_A} (1-\theta)^{\eta_L}$. Inserting these output functions into the general system (13) to (18), the dynamic system in scale-adjusted variables may be expressed as follows.

$$\dot{k} = \alpha_F\, a^{\sigma_A} \theta^{\sigma_L} k^{\sigma_K} - c - \delta_K\, k - \beta_K\, n\, k \tag{22}$$

$$\dot{a} = \alpha_J\, a^{\eta_A} (1-\theta)^{\eta_L} - \beta_A\, n\, a \tag{23}$$

$$\dot{c} = \frac{c}{\gamma}\left[\frac{{\sigma_K}^2 \alpha_F a^{\sigma_A} \theta^{\sigma_L} k^{\sigma_K}}{k} - \delta_K - \rho - (1-\gamma)\, n\right] - \beta_K\, n\, c \tag{24}$$

$$\dot{v}_a = v_a\left[\frac{{\sigma_K}^2 \alpha_F a^{\sigma_A} \theta^{\sigma_L} k^{\sigma_K}}{k} - \delta_K - (\beta_K - \beta_A)\, n\right] - \frac{(1-\sigma_K)\, \sigma_K \alpha_F a^{\sigma_A} \theta^{\sigma_L} k^{\sigma_K}}{a} \tag{25}$$

[11] The social formulation of this model was introduced into the literature by Eicher and Turnovsky (1999b).

[12] Since both sectors are characterised by Cobb-Douglas technologies, the preceding conditions are also sufficient for balanced growth (Eicher and Turnovsky, 1999a, p. 402 and p. 404).

$$\sigma_L \, \alpha_F \, a^{\sigma_A} \theta^{\sigma_L - 1} k^{\sigma_K} = v_a \, \eta_L^p \, \alpha_J a^{\eta_A} (1-\theta)^{\eta_L - 1} \tag{26}$$

3.2.2. Discussion

3.2.2.1. Rate of convergence and key economic ratios

The baseline set of parameters underlying the numerical computations performed in this section is shown in Table 1. The parameters are essentially identical to those used in previous exercises (e.g., Prescott, 1986; Lucas 1988; King and Rebelo, 1993; Ortigueira and Santos, 1997; Jones 1995b; Eicher and Turnovsky, 2001). The final output sector exhibits constant returns to scale in the private inputs (labour and physical capital) but increasing returns to scale in all three factors (including technology). Similarly, the R&D sector is subject to decreasing returns in labour but shows mildly increasing returns with respect to labour and technology.

TABLE 1

TABLE 2

Let us now turn to the quantitative implications resulting from the theoretical experiment conducted here. Table 2 reports several growth rates ($\widehat{Y/L}$, $\widehat{A/L}$), key economic ratios ($\widetilde{Y/K}$, $\widetilde{C/Y}$), the stationary value of scale-adjusted output ($\tilde{y}$), the labour allocation variable ($\tilde{\theta}$) and the asymptotic rate of convergence of Y/L ($\tilde{\psi}_{Y/L}$) for the market solution and for the social solution. All values lie within highly plausible ranges. Several points are especially worth being noted.

First, the growth rates of Y/L (which equals the growth rate of K/L) and A/L are identical for the market and the social solution as suggested above. Since the returns to scale in the final output sector exceed those in the R&D sector, the growth rate of Y/L (K/L) lies above the growth rate of A/L. The output-capital ratio is significantly higher for the market solution compared to the social solution. The economic explanation for this divergence lies in the fact that the market economy saves less ($\widetilde{C/Y}_M > \widetilde{C/Y}_S$) due to the downward biased interest rate.

Second, the decentral share of labour devoted to R&D exceeds the socially optimal share, i.e. $1-\tilde{\theta}_M > 1-\tilde{\theta}_S$. Jones (1995a) shows that there are three effects at work in non-scale R&D-based growth models which induce the decentral share to deviate from the socially optimal share. (1) If $\eta_A > 0$, then the intertemporal spill-over effect of technological knowledge causes the economy to underinvestment in R&D; (2) provided that $\eta_L < 1$, there are negative (intratemporal) externalities due to the duplication of research inducing too much resources to be allocated to R&D; (3) the monopoly mark-up over marginal cost in the sale of intermediate goods induces too little labour to be devoted to R&D. Since $\eta_L = 0.6$, the second effect clearly causes the decentral economy to overinvest in R&D. In addition, for the baseline set of parameters the second effect seems to dominate the first and the third effect.

Third, the level of the balanced growth path, as indicated by $\tilde{y}$, differs drastically between the market and the social solution. Since the balanced growth path in terms of per capita output is given by $\tilde{Y}(t)/L(t) = \tilde{y}\, e^{(\beta_K - 1)nt}$ [where we have set $L(0)=1$], $\tilde{y}$ immediately shows the level of the balanced growth path. Scale-adjusted output amounts to 41 units (expressed in units of final output) for the market solution as opposed to 67 units in the case of the social solution. Put differently, the level of the socially optimal balanced growth path exceeds that of the decentral balanced growth path by 64 %.

Fourth, Table 2 shows that per capita income asymptotically converges at a rate of about 0.7 % in the case of the social solution and at a lower rate of 0.5 % provided that the market solution is considered. Both rates are surprisingly low, that is the economy converges very slowly. The rates of convergence can be easily transformed into half-life times by use of the formula $\psi_x = -Log(0.5)/t$. The social solution implies a half-life time of per capita income amounting to 102 years, while the market solution shows a half-life time of 137 years.[13] The economic intuition for this difference is as follows. Along the transition, preferences (as expressed by the Keynes-Ramsey rule) additionally influence the dynamics of the economy. Moreover, it has been stated above that the decentral real interest rate is biased downwards due to the market power of intermediate goods producers. Therefore, the decentral economy gives weaker incentives to accumulate resources and speed up growth along the transition. As a result, the gap between the current state of the economy and the balanced growth path is closed more slowly.

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[13] Not surprisingly, a similar pattern is found for per capita capital and per capita technology (not reported).

Altogether, these results provide strong arguments in favour of transitional dynamics vis-à-vis balanced-growth dynamics. Since the political system is far from being perfect, even in developed countries, positive analyses of the growth process should be based on the market economy model. In this respect it is important to notice that the market solution converges even slower.

3.2.2.2. Sensitivity analysis

What about the robustness of the preceding findings? It is clear that the rates of convergence shown in Table 2 are calculated for a specific set of parameters and are valid for one point in the parameter space only. On the one hand, this procedure appears justified since the rate of convergence cannot be derived analytically due to the complexity of the system under study. On the other hand, this procedure clearly necessitates a sensitivity analysis.

The local rate of convergence of per capita income is given by $\widetilde{\psi}_{Y/L} = -\lambda_1(\sigma_A + \sigma_L + \sigma_K) - (\beta_K - 1)n$, where λ_1 denotes the dominant of the stable eigenvalues (i.e. $\lambda_2 < \lambda_1 < 0$) and $\beta_K = \dfrac{\sigma_L(1-\eta_A) + \eta_L\,\sigma_A}{(1-\eta_A)(1-\sigma_K) - \eta_K\,\sigma_A}$.[14] If, for example, σ_A varies, then the rate of convergence changes according to $\dfrac{\partial \widetilde{\psi}_{Y/L}}{\partial \sigma_A} = -\dfrac{\partial \lambda_1}{\partial \sigma_A}\sigma_A - \lambda_1 - \dfrac{\partial \beta_K}{\partial \sigma_A} n$. Each of the components of the preceding partial derivative is known analytically and small in magnitude (i.e. of second order). However, the magnitude of $\dfrac{\partial \lambda_1}{\partial \sigma_A}$ is unknown since we have no analytic expression for λ_1.

In order to assess the sensitivity of λ_1 with respect to parameters changes, we evaluate λ_1 for successive values of the parameter under consideration. This procedure varies one of

[14] At this point it is helpful to realise that the generic form of the stable solution to the linearised problem is given by $k = v_{11}B_1e^{\lambda_1 t} + v_{12}B_2e^{\lambda_2 t} + \tilde{k}$, $a = v_{21}B_1e^{\lambda_1 t} + v_{22}B_2e^{\lambda_2 t} + \tilde{a}$, $\theta = v_{31}B_1e^{\lambda_1 t} + v_{32}B_2e^{\lambda_2 t} + \tilde{\theta}$, where λ_1 and λ_2 denote the stable roots ($\lambda_2 < \lambda_1 < 0$), B_1 and B_2 arbitrary constants (depending on the shock under consideration) and v_{ij} the elements of the eigenvectors associated with the corresponding stable root λ_j. Moreover, the deviation of y from $\tilde{y}$ may be expressed to read $y - \tilde{y} = \alpha_F(a-\tilde{a})^{\sigma_A}(\theta-\tilde{\theta})^{\sigma_L}(k-\tilde{k})^{\sigma_K}$. Since $\lambda_2 < \lambda_1 < 0$, the asymptotic rate of convergence is determined by λ_1. Consequently, in the limit the gap $y - \tilde{y}$ evolves according to $y - \tilde{y} = \alpha_F\left(v_{11}B_1\right)^{\sigma_K}\left(v_{21}B_1\right)^{\sigma_A}\left(v_{31}B_1\right)^{\sigma_L} e^{\lambda_1(\sigma_A+\sigma_L+\sigma_K)t}$. The asymptotic rate of convergence of y reads $\tilde{\psi}_y = -\lambda_1(\sigma_A + \sigma_L + \sigma_K)$ and from $y := Y/L^{\beta_K}$ we get $\widetilde{\psi}_{Y/L} = -\lambda_1(\sigma_A + \sigma_L + \sigma_K) - (\beta_K - 1)n$.

the parameters holding the others fixed, i.e. the baseline set of parameters is used as an anchor. This allows us to assess the robustness of the rate of convergence and to reach more general conclusions on the speed of convergence.

The results of this experiment are displayed in Figure 1 to Figure 3, which depict the relationship between the smaller of the two negative eigenvalues, in absolute terms, and the different parameter values both for the market solution (dashed line) and for the social solution (solid line). Notice that the vertical line marks the respective parameter value in the baseline set.[15] Since the smaller eigenvalue, in absolute terms, dominates the larger one if time proceeds, the former is labelled dominant eigenvalue. The parameters have been grouped into three categories: (1) Final output technology parameters (Figure 1); (2) R&D technology parameters (Figure 2) and (3) preference parameters together with the population growth rate (Figure 3). At least three important conclusions can be drawn from this sensitivity analysis.

FIGURE 1

FIGURE 2

FIGURE 3

First, the most important result lies in the fact that the eigenvalues do not vary substantially in response to parameter changes; the strongest impact comes from the population growth rate displayed in Figure 3 (c). This proposition applies to both the decentral and the social dominant eigenvalue. Therefore, the result of slow convergence speeds is robust with respect to parameter changes.

Second, the dominant eigenvalue for the market solution lies strictly below (in absolute terms) the corresponding eigenvalue for the social solution. This implies that the asymptotic rate of convergence of the market economy is strictly smaller than the asymptotic rate of convergence of the social economy. Put differently, the market economy converges slower than the socially controlled economy.

[15] Parameter restrictions have not been taken into account. As a result, some combinations of parameters might lie outside the economically relevant range. This does not, however, limit the results of the sensitivity analysis.

Third, Figure 1 to Figure 3 provide information on the relationship between the rate of convergence and the different economic parameters. On the one hand, the rate of convergence appears independent of total factor productivity in both sectors, i.e. independent of the scale factors α_F and α_J. This finding is in line with the results on the rate of convergence for the neoclassical model (Barro and Sala-i-Martin, 1992, pp. 225/226). On the other hand, all other parameters of the model seem to affect the rate convergence. In contrast, Ortigueira and Santos (1997, p. 386) find that the rate of convergence is independent of preference parameters for the investment-based endogenous growth model of the Uzawa-Lucas type.

Finally, a caveat is clearly indicated. What has been discussed so far is the local speed of convergence around the balanced growth path. However, the local rate of convergence need not be a valid approximation of the global convergence behaviour. Especially if we are interested in out-of-balanced-growth dynamics, we should have a closer look at the global rate of convergence. The next section addresses this shortcoming by visually characterising the time path of the instantaneous rates of convergence.[16]

3.2.2.3. Illustration of transitional dynamics

The qualitative aspects of the transition process are illustrated. Instead of reporting the results of numerous simulations, the characteristic properties of the adjustment dynamics are discussed. The potential characteristics of the adjustment processes are demonstrated by two simulations.[17] In the first case, the economy converges from below, while in the second case the economy converges from above the stationary solution.

Figure 4 highlights several interesting aspects of the first adjustment process. The source of transitional dynamics is an exogenous permanent technological shock in the production of final output. More specifically, α_F is assumed to increase from 0.5 to 1. Plot (a) shows the trajectory in (k, a)-space and illustrates that the economy converges (globally) from below its stationary equilibrium. Plot (b) depicts the time path of the share of labour devoted to the production of final output. It can be observed that this variable initially decreases, reaches a minimum and subsequently converges to its long-run level. This pattern is largely mirrored by the time path of the growth rate of per capita technology as displayed in

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[16] Ortigueira and Santos (1997) show that the local rate of convergence is indeed a valid approximation of the global convergence behaviour.

[17] The underlying differential equation system is approximated numerically by using backward integration; see Brunner and Strulik (2002) for details on this procedure.

plot (f). The time paths of the rates of convergence of scale-adjusted output (ψ_y, dashed line) and of scale-adjusted capital (ψ_k, solid line) are portrayed in plot (c). The rates of convergence are variable over time. They pass through the positive as well as the negative range. More specifically, the time paths exhibit a singularity indicating that point in time at which the respective scale-adjusted variable overshoots its long-run equilibrium level.[18] In the limit the rates of convergence approach their respective long-run equilibrium level given by $\widetilde{\psi}_y = -\lambda_1(\sigma_A + \sigma_L + \sigma_K)$ and $\widetilde{\psi}_k = -\lambda_1$. Plot (d) shows the time paths of the growth rates of per capita output (dashed line) and per capita capital (solid line). The plot shows that both growth rates decrease monotonically and converge to their long-run value. Since the growth rate of per capita output is positive and decreasing this adjustment process implies conditional β-convergence. Finally, plot (e) displays the time path of the rate of convergence of scale-adjusted technology. This time path obeys a singularity as well, indicating that scale-adjusted technology overshoots its long-run level.

FIGURE 4

The second adjustment process is illustrated in Figure 5. The source of transitional dynamics in this case is a permanent decrease in α_F from 1.5 to 1. Plot (a) shows the adjustment trajectory in (k,a)-space. In this case, the transition path approaches the stationary solution from above. It should be noticed that a decrease in scale-adjusted variables does not necessarily imply a decrease of the respective variable measured along original scale. It merely means that the corresponding original variable grows at a rate which lies below its balanced-growth rate. Plot (b) demonstrates that the share of labour devoted to final output production decreases significantly and converges to its long-run equilibrium level. The time profiles of the rates of convergence of scale-adjusted output (ψ_y, dashed line) and of scale-adjusted capital (ψ_k, solid line) are illustrated in Plot (c). The singularities immediately indicate an overshooting of the respective variables. Figure 5 (d) shows the time paths of the growth rates of per capita output and per capita capital. It can be observed that the growth rate of Y/L and K/L increase along the transition to their balanced growth levels. Since the growth rate of Y/L is positive and increases along the transition, this adjustment path implies

[18] In fact, since the state variables overshoot their long-run levels, the trajectory under study locally convergence from above its stationary state. Globally, however, the economy converges from below.

conditional β-divergence. The time path of the rate of convergence of scale-adjusted technology obeys a singularity as well, as displayed in plot (e). Finally, plot (f) shows the time path of the growth rate of per capita technology largely mirroring the time path of the labour allocation variable.

FIGURE 5

To sum up, the basic non-scale R&D-based model shows a wide range of possible adjustment dynamics. Non-monotonic adjustment paths and variable convergence rates seem to be an intrinsic element of the out-of-balanced-growth dynamics. Extensive simulation exercises have shown that the qualitative features reported above appear robust with respect to changes in the underlying parameters. The findings confirm the results of Eicher and Turnovsky (1999b, 2001), who found similar characteristics of the transition process for the social solution.

3.3. The generalised non-scale model of R&D-based growth

3.3.1. The model

The generalised model of R&D-based growth postulates that physical capital is also productive in the R&D sector. The production side of the economy in terms of aggregate capital is given as follows.[19]

$$Y = F(.) = \alpha_F\, A^{\sigma_A} (\theta L)^{\sigma_L} (\phi K)^{\sigma_K} \quad \text{with} \quad \sigma_A, \sigma_L, \sigma_K > 0\,;\; \sigma_L + \sigma_K = 1 \tag{27}$$

$$\dot{A} = J(.) = \alpha_J\, A^{\eta_A} [(1-\theta) L]^{\eta_L} [(1-\phi) K]^{\eta_K}$$
$$\text{with} \quad \eta_A, \eta_L, \eta_K > 0\,;\; \eta_L + \eta_K \le 1 \;\; (\eta_L^p + \eta_K^p = 1,\; \eta_L^e + \eta_K^e \le 0) \tag{28}$$

$$\dot{L} = n\, L \tag{29}$$

The balanced-growth rates turn out to read $\hat{K} = \dfrac{[\sigma_L (1-\eta_A) + \eta_L\, \sigma_A]\, n}{(1-\eta_A)(1-\sigma_K) - \eta_K\, \sigma_A} = \beta_K\, n$ and $\hat{A} = \dfrac{[\eta_L (1-\sigma_K) + \eta_K\, \sigma_L]\, n}{(1-\eta_A)(1-\sigma_K) - \eta_K\, \sigma_A} = \beta_A\, n$. Necessary and sufficient conditions for positive per

[19] The social formulation of this model was introduced into the literature by Eicher and Turnovsky (1999a).

capita growth are $(1-\eta_A)(1-\sigma_K)-\eta_K\,\sigma_A>0$ and $\sigma_K<1$. Since both sectors are characterised by Cobb-Douglas technologies, the preceding conditions are also sufficient for balanced growth (Eicher and Turnovsky, 1999a, p. 402 and p. 404).

Let us now turn to the dynamic system in scale-adjusted variables. From (27), (28) together with $y:=Y/L^{\beta_K}$, $k:=K/L^{\beta_K}$, $c:=C/L^{\beta_K}$, $a:=A/L^{\beta_A}$, $j:=J/L^{\beta_A}$, $v_a:=v/L^{\beta_K-\beta_A}$ and considering $\beta_K=\dfrac{\sigma_L\,(1-\eta_A)+\eta_L\,\sigma_A}{(1-\eta_A)(1-\sigma_K)-\eta_K\,\sigma_A}$ and $\beta_A=\dfrac{\eta_L\,(1-\sigma_K)+\eta_K\,\sigma_L}{(1-\eta_A)(1-\sigma_K)-\eta_K\,\sigma_A}$ we can derive the production functions in scale-adjusted variables to read $y=\alpha_F\,a^{\sigma_A}\,\theta^{\sigma_L}\left(\phi\,k\right)^{\sigma_K}$ and $j=\alpha_J a^{\eta_A}\left(1-\theta\right)^{\eta_L}\left[\left(1-\phi\right)k\right]^{\eta_K}$. Inserting these output functions into the general system (13) to (18), the dynamic system in scale-adjusted variables may be expressed as follows.

$$\dot{k}=\alpha_F\,a^{\sigma_A}\,\theta^{\sigma_L}\left(\phi\,k\right)^{\sigma_K}-c-\delta_K\,k-\beta_K\,n\,k \tag{30}$$

$$\dot{a}=\alpha_J a^{\eta_A}\left(1-\theta\right)^{\eta_L}\left[\left(1-\phi\right)k\right]^{\eta_K}-\beta_A\,n\,a \tag{31}$$

$$\dot{c}=\frac{c}{\gamma}\left[\frac{\sigma_K{}^2\alpha_F\,a^{\sigma_A}\,\theta^{\sigma_L}\left(\phi\,k\right)^{\sigma_K}}{\phi\,k}-\delta_K-\rho-(1-\gamma)\,n\right]-\beta_K\,n\,c \tag{32}$$

$$\dot{v}_a=v_a\left[\frac{\sigma_K{}^2\alpha_F\,a^{\sigma_A}\,\theta^{\sigma_L}\left(\phi\,k\right)^{\sigma_K}}{\phi\,k}-\delta_K-(\beta_K-\beta_A)\,n\right]-\frac{\left(1-\sigma_K\right)\sigma_K\,\alpha_F\,a^{\sigma_A}\,\theta^{\sigma_L}\left(\phi\,k\right)^{\sigma_K}}{\phi\,a} \tag{33}$$

$$\sigma_L\,\alpha_F\,a^{\sigma_A}\,\theta^{\sigma_L-1}\left(\phi\,k\right)^{\sigma_K}=v_a\,\eta_L^p\alpha_J a^{\eta_A}\left(1-\theta\right)^{\eta_L-1}\left(1-\phi\right)^{\eta_K}k^{\eta_K} \tag{34}$$

$$\sigma_K\,\alpha_F\,a^{\sigma_A}\,\theta^{\sigma_L}\phi^{\sigma_K-1}k^{\sigma_K}=v_a\,\eta_K^p\,\alpha_J a^{\eta_A}\left(1-\theta\right)^{\eta_L}\left(1-\phi\right)^{\eta_K-1}k^{\eta_K} \tag{35}$$

3.3.2. Discussion

3.3.2.1. Rate of convergence and key economic ratios

The baseline set of parameters underlying the study of this model is shown in Table 3. This set of parameters is similar to the one shown in Table 1. However, there are a number of restrictions, which force us to deviate from the initial baseline set of parameters. At first, we assume that the elasticities of the private inputs in R&D are $\eta_L=0.5$ and $\eta_K=0.4$ implying mildly decreasing returns to scale in R&D in the private inputs. To obtain constant returns to scale at the level of the firm we set $\eta_L^e = -0.06$ and $\eta_K^e = -0.04$; note that this yields $\eta_L^p + \eta_K^p = 1$. Second, in the course of the derivation of the general dynamic system it has been assumed that intermediate goods producers have no incentive to differentiate their supply price vis-à-vis their two groups of customers (final output and R&D producers). This simplifying assumption requires that $\sigma_K = \eta_K^p$ and, hence, we set $\sigma_K = 0.44$. In order to obtain constant returns to scale in the final output sector we further assume $\sigma_L = 0.56$. Third, for reasons of comparability we choose $\sigma_A = 0.12$ to achieve a growth rate of per capita income similar to the one obtained within the previous exercise.

TABLE 3

TABLE 4

Due to the model calibration, the balanced growth rate of per capita income shown in Table 4 is nearly identical to the one resulting from the basic non-scale model. The balanced growth rate of technology is significantly higher compared to the previous model. This is due to the fact that capital is productive in R&D in addition to labour and technology. The output-capital ratio along the balanced growth path is significantly lower compared to the basic non-scale model. This result is not surprising since more capital is accumulated in this economy due to the fact that capital has a second productive use. Similarly, the consumption-output ratio is slightly lower along the balanced growth path compared to the basic non-scale model. This signifies that a larger ratio of current output is saved and used for capital accumulation. Considering the allocation variables $\tilde{\theta}$ and $\tilde{\phi}$ reveals very plausible values. These variables indicate that, according to the model under study, 13 % of the labour force and the capital stock are allocated to R&D, while the rest is engaged in final output production. The fact that both $\tilde{\theta}$ and $\tilde{\phi}$ are identical is not surprising since from (17) and (18) it immediately follows that $\frac{\sigma_L}{\eta_L^p}\frac{1-\theta}{\theta} = \frac{\sigma_K}{\eta_K^p}\frac{1-\phi}{\phi}$. Moreover, since $\sigma_K = \eta_K^p$ by assumption and $\sigma_L = \eta_L^p$ by choice of parameters it follows that $\theta = \phi$. This relation holds true along the transition path as well as along the balanced growth path. Finally, the (asymptotic) rate of convergence of per capita income turns out to be 2.7 %. The implied half-life time is around

25 years.[20] Two points should be observed. First, the generalised non-scale R&D-based model contains one further mechanism of intertemporal consumption substitution, which consists in the allocation of capital goods to R&D. It should, therefore, not be surprising that the implied rate of convergence is considerably higher. Second, different parameter restrictions applying to the specific models, however, restrict comparability.

3.3.2.2. Illustration of transitional dynamics

Let us now turn to the qualitative convergence implications. As before, the source of transitional dynamics is an exogenous permanent technological shock in the production of final output. More specifically, α_F is assumed to decrease from 1.5 to 1. Figure 6 (a) shows the adjustment trajectory in (k,a)-space. The striking feature here is the observation of severalfold over- and undershooting, which represents an additional characteristic of the adjustment process. This phenomenon can be recognised even more clearly by inspecting the time paths of the (instantaneous) rates of convergence. Specifically, each singularity in these time paths indicates one over- or undershooting of the underlying variable. Scale-adjusted capital and scale-adjusted technology over- and undershoots several times [plot (c) and plot (d)]. As a result, scale-adjusted output over- and undershoots several times as well [plot (b)]. These plots additionally show that the rate of convergence is variable along the transition path. Finally, the labour allocation variable (θ) and the capital allocation variable (ϕ) are characterised by highly non-monotonic adjustments as well. Figure 6 (e) and (f) shows the time paths of these variables. As before we observe a succession of over- and undershooting. In the limit monotonic converges is obtained.

FIGURE 6

Since we cannot observe scale-adjusted variables in the real world it is clearly desirable to know how this pattern of adjustment translates into original variables or transformations of the original variables, which can be observed empirically. Figure 7 shows the time path of the growth rate of per capita output [plot (a)] together with the time path of (logarithmic) per capita output [plot (b)]. The dashed line in plot (b) represents the balanced growth path in

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[20] Observe that from $y = \alpha_F \, a^{\sigma_A} \, \theta^{\sigma_L} \, (\phi \, k)^{\sigma_K}$ it follows that the rate of convergence of per capita income is given by $\widetilde{\psi}_{Y/L} = -\lambda_1(\sigma_A + \sigma_L + 2\sigma_K) - (\beta_K - 1)n$ (compare to footnote 14).

terms of (logarithmic) per capita output and is calculated by using the formula $\tilde{Y}(t)/L(t) = \tilde{y}\, e^{(\beta_K - 1)nt}$, where we have set $L(0)=1$. Corresponding plots are displayed for capital per capita [plot (c) and (d)] and technology per capita [plot (e) and (f)].

FIGURE 7

The analysis reveals several interesting points. First, the succession of over- and undershooting in scale-adjusted variables translates into cyclical movements in the growth rates of the respective per capita variable. Similarly, the growth path of the respective per capita variables fluctuates around its balanced growth path. It should be noted that we observe only some of the fluctuations, while the remaining are to small to be recognised by inspection. Again, in the limit monotonic convergence is obtained. Second, since the adjustment process takes decades to approach reasonably close to its balanced growth path these fluctuations should be interpreted as growth cycles. Third, the cycles result from the comparably high dimension and high degree of non-linearity of the underlying dynamic system. It is important to stress that the cycles are not caused by complex eigenvalues and trigonometric components in the solution. For this reason the cycles finally come to an end and monotonic convergence is obtained.[21] Therefore, the phenomenon observed here can be labelled as non-complex growth cycles. Fourth, the non-monotonic adjustments and the resulting time paths of the instantaneous rate of convergence [Figure 7 (b), (c) and (d)] demonstrate that the notion of exclusive (conditional) β-convergence or of exclusive (conditional) β-divergence along the transition to the balanced growth equilibrium might be too simple. The model studied here shows that both phenomena can occur along the transition. In this case, the concept of the rate of convergence does not appear useful to determine the speed at which the economy converges to its balanced-growth path. A more appropriate concept should measure how fast the amplitude of the growth cycles vanishes.[22]

What causes the cyclical adjustments that arise along the transition to the balanced growth path? At first, consider the technical conditions for non-monotonic adjustment dynamics. It is well-known that the (stable) solution to a linearised dynamic system with two

[21] In contrast, trigonometric adjustment processes are characterised by never ending fluctuations, although the amplitude might tend to zero with time approaching infinity.

[22] In addition, it is unclear whether the standard methods used to estimate the rate of convergence empirically produce spurious results in the presence of growth cycles. The findings might be highly sensitive with respect to the time period under study.

negative roots (describing a two-dimensional stable manifold) can obey non-monotonic adjustments. However, the maximum degree of non-monotonicity in this case is one-time overshooting (remember that no complex eigenvalues are involved).[23] Therefore, the cyclical movements shown in Figure 7 can only occur along the non-linear two-dimensional stable manifold. In the next place, let us sketch the economic intuition behind this pattern of development. This is best described using Figure 8 which shows the time paths of $k(t)$, $a(t)$, $\theta(t)$ and $v_a(t)$. The starting point is a permanent decrease in the exogenous technology parameter α_F. As a result, the allocation variables (θ and ϕ) obey a downward jump (not shown in Figure 8). This means that private resources (capital and labour) are discontinuously reallocated from final output (capital) production to R&D production. In the wake of this reallocation scale-adjusted capital falls and scale-adjusted technology rises [Figure 8 (a) and (b)]. Observe that the correlation between both variables (k and a) is perfectly negative. This is due to the fact that both θ and ϕ move together as can be readily shown analytically by eliminating v_a from (34) and (35). As the ratio of k and a falls, the relative price of a in terms of k (given by v_a) decreases as well [Figure 8 (d)]. As a result, resources are reallocated from the R&D sector to the final output sector, that is θ and ϕ gradually rise. But as long as θ and ϕ are below their long-run values, k continues to decrease and a continues to increase. At that point in time for which $\theta = \tilde{\theta}$ and $\phi = \tilde{\phi}$, it holds true that $\dot{k} = 0$ and $\dot{a} = 0$. However, the system is not yet in its long-run equilibrium since $k < \tilde{k}$ and $a > \tilde{a}$. Therefore, the allocation variables must increase further, thereby overshooting their long-run levels. At this stage $\theta > \tilde{\theta}$ and $\phi > \tilde{\phi}$ causes k to increase and a to decrease. As a result, the price of a in terms of k starts to increase. This development at first reduces the increase in the allocation variables and finally induces the allocation variables to turn downwards. From this point the process continues with the sign of the movements reversed. Taken together the cyclical fluctuations result from the interplay between instantaneously adjusting control variables and slowly adjusting state variables. An important feedback mechanism is based on the adjustment of the relative price of the two state variables.[24]

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[23] This is demonstrated by Eicher and Turnovsky (2001) using a qualitative reasoning and shown analytically by Bovenberg and Smulders (1996).

[24] In order to solve the underlying dynamic system numerically the routine NDSolve of Mathematica was employed. This routine switches between a non-stiff Adams method and a stiff Gear method. The absolute error (AccuracyGoal) as well as the relative error (PrecisionGoal) were set to 10^{-10}. In addition, a Runge-Kutta (Fehlberg order 4-5) method was employed. Although the two methods give slightly different numeric solutions,

4. Summary and conclusion

Viewed from a methodological perspective, growth theorists possess three basic approaches to explain the diverse growth experiences observable in the real world. First, the unique-balanced-growth-equilibrium approach relies on unique balanced growth paths and requires parameter heterogeneity to explain diverse growth experiences. Second, the multiple-balanced-growth-equilibrium approach emphasises the importance of multiple balanced growth paths and assumes different initial conditions to explain diverse growth experiences. Both approaches interpret real-world economic growth exclusively as balanced growth equilibria. Third, the transitional-dynamics approach considers real-world growth to mainly represent transitional dynamics towards (unique) balanced growth paths. This approach can be based on different initial conditions as well as on parameter heterogeneity. Although there are a number of papers investigating transitional dynamics within endogenous growth models in the meantime, this approach appears largely under-utilised. The paper in hand takes important steps in this direction. The transitional dynamics implications within the probably most important strand of growth models are investigated comprehensively. More specifically, the basic non-scale R&D-based model and the generalised non-scale R&D-based model of endogenous growth are employed. The focus is on the market solution albeit the social solution is also considered.

The question for the relative importance of transitional vis-à-vis balanced-growth dynamics can be answered as follows. If we take R&D-based growth models seriously, then we find that transitional dynamics play an important role. The basic non-scale model yields surprisingly low rates which range from 0.5 % to 0.7%. These values imply half-life times of about 102 years to 137 years.[25] In addition, it is shown that the market solution converges more slowly compared to the social solution. Extensive sensitivity analyses shows that the results are robust with respect to parameter changes. The generalised non-scale model yields a considerable higher rate of 2.7 % implying a half-life time of 25 years.[26] Even this considerably higher speed of convergence indicates that the balanced growth path does not

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the qualitative characteristics remained unchanged. In addition, the accuracy of the resulting numeric solution to a differential equation $\dot{x}(t) = G[x(t)]$ can be described by the residual defined as $res(t) := G[x(t)] - \dot{x}(t)$. Using the numeric solution for $x(t)$ this residual has been explicitly determined. It turns out that this residual is of order 10^{-8} at maximum. Moreover, the qualitative characteristics of the numeric solution remained unchanged for different sets of parameters.

[25] It should be noted that Jones (1995a) found even higher half-life times for a very similar model.

[26] A direct comparison between the two kinds of models is difficult due to additional parameter restriction for the more general model resulting in a different set of parameters.

tell the complete story about economic growth. Moreover, it is very important to notice that the results on the speed of convergence are probably biased downwards, i.e. the "true" half-life times can be expected to be considerably higher. This is due to the fact that standard growth models abstract from resource reallocation costs. The findings, therefore, provide strong arguments in favour of transitional dynamics as opposed to balanced-growth dynamics.

The analyses conducted in this paper reveal a number of important qualitative convergence implications. The basic non-scale R&D-based model is able to reproduce an overshooting or an undershooting of the state variables. The findings confirm the results of Eicher and Turnovsky (1999b, 2001), who found similar characteristics of the transition process for the social solution. The generalised non-scale R&D-based model, which additionally views capital to be productive in R&D, shows a succession of over- and undershootings of the endogenous variables along the transition path. While this observation applies to scale-adjusted variables, this phenomenon translates into cyclical movements of the original variables. More specifically, the growth path of per capita output turns out to fluctuate around its balanced growth path. This pattern of development is accompanied by fluctuations in the growth rate of per capita output around its long-run value. Since the adjustment process takes decades to approach reasonably close to its balanced growth path, these fluctuations should be interpreted as growth cycles. No complex eigenvalues and trigonometric solutions are involved and, hence, the analyses shows an alternative route to growth cycles. The cyclical adjustments result from the comparably high dimension and high degree of non-linearity of the underlying dynamic system. This phenomenon is therefore labelled as non-complex growth cycles. Since the real world is most probably best described by highly dimensional and highly non-linear dynamic systems, we should learn from the theoretical experiments conducted in this paper that cyclical adjustment processes represent the norm rather than a special case. Moreover, the analysis demonstrates that the specific parameter constellations leading to complex eigenvalues are not necessary to give rise to cyclical movements.

The results presented in this paper further contain some important policy implications. First, as Jones (1995a) has demonstrated, non-scale growth models imply that policy is ineffective with respect to the balanced growth rate. In addition, policies controlling for the balanced growth rate are simply inappropriate since the decentral and the social balanced growth rate coincide. Second, as Jones (1995a) and Turnovsky (2000) stress, policy measures can nonetheless influence the rate of growth along the transition path. Their accumulated effects translate into higher levels of the balanced growth path. In this respect the relative importance of transitional dynamics vis-à-vis balanced growth dynamics is once more of crucial importance.

Finally, the paper points to some interesting questions for future research. The models which are employed in growth theory typically assume that resources can be shifted instantaneously from one sector to another. On the one hand, this assumption greatly simplifies the analyses. On the other hand, however, it should be clear that this assumption might be crucial with respect to the relative-importance question under study. The explicit consideration of reallocation costs can be expected to significantly increase the time span which is required to adjust once more closely to the balanced growth path. Structural changes do represent an intrinsic element of real world economic dynamics. In order to analyse the process of structural adjustments, resource reallocation costs should therefore be incorporated into the analyses of transitional dynamics.

5. Appendix

5.1. The household's optimisation problem (market economy)

The dynamic problem together with its solution is summarised by the following set of equations; the Hamiltonian is given in present-value form.

$$\max_{\{C/L\}} \int_0^\infty \frac{(C/L)^{1-\gamma}-1}{1-\gamma} e^{-\rho t}\, dt \tag{A.1}$$

$$s.t.\;\; \dot{K} = r_n K + wL + A\pi - v\dot{A} - C\,,\;\; K(0) > 0 \tag{A.2}$$

$$H(C/L, K, \lambda) := \frac{(C/L)^{1-\gamma}-1}{1-\gamma} + \lambda\left[r_n K + wL + A\pi - v\dot{A} - C\right] \tag{A.3}$$

$$H_{C/L} = (C/L)^{-\gamma} - \lambda L = 0 \;\;\Leftrightarrow\;\; C^{-\gamma} = \lambda L^{1-\gamma} \tag{A.4}$$

$$\dot{\lambda} = -H_K + \rho\lambda = -\lambda r_n + \rho\lambda \;\;\Leftrightarrow\;\; \frac{\dot{\lambda}}{\lambda} = \rho - r_n \tag{A.5}$$

$$\frac{\dot{C}}{C} = \frac{1}{\gamma}\left[r_n - \rho - (1-\gamma)n\right] \tag{A.6}$$

$$\lim_{t\to\infty} e^{-\rho t}\lambda(t)K(t) = 0\,, \tag{A.7}$$

where $\pi = \dfrac{G(\theta L, \phi K, A)K}{\varepsilon q A}$, $r = \dfrac{\varepsilon - 1}{\varepsilon}\dfrac{G(\theta L, \phi K, A)}{q}$, $w = F_{\theta L}(.)$, v is given by (6) together with $v(0) > 0$, $\dot{A}$ by (4), A by (4) together with $A(0) > 0$ and L by $\dot{L} = nL$ together with $L(0) > 0$. Provided that the Hamiltonian is jointly concave in the control and the state variable (Mangasarian sufficiency conditions) or that the maximised Hamiltonian is concave in the state variable (Arrow sufficiency conditions), the necessary conditions are also sufficient.[27] The transversality condition demands for the following inequality constraint to be met $-\rho + \lim_{t\to\infty}\hat{\lambda} + \lim_{t\to\infty}\hat{K} < 0$.

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[27] For details on sufficiency conditions within optimal control theory see Kamien and Schwartz (1981, part II section 3 and section 15).

5.2. Balanced growth rates (market economy)

Following Eicher and Turnovsky (1999a) we employ the auxiliary assumption that $\hat{Y} = \hat{K}$ in the long-run, which is in line with empirical evidence (Romer, 1989). From $\hat{K} = Y/K - \delta_K - C/K$ it then follows that balanced growth further requires $\hat{K} = \hat{C}$. The balanced growth rates of K and A can be derived from $\frac{d}{dt}\frac{F(.)}{K} = 0$ and $\frac{d}{dt}\frac{J(.)}{A} = 0$ by noting that the allocation variables are constant.[28]

$$\frac{\left[F_A(.)\dot{A} + F_L(.)\dot{L} + F_K(.)\dot{K}\right]K - F(.)\dot{K}}{K^2} = 0 \tag{A.8}$$

After some straightforward manipulations the preceding equation can be expressed as follows.

$$\sigma_A \hat{A} + \sigma_L \hat{L} + \sigma_K \hat{K} - \hat{K} = 0 \tag{A.9}$$

where $\sigma_x := \frac{F_x(.)x}{F(.)}$ for $x = A, K, L$.

Similarly, balanced growth of A requires $\frac{d}{dt}\frac{J(.)}{A} = 0$. Carrying out this instruction yields.

$$\frac{\left[J_A(.)\dot{A} + J_L(.)\dot{L} + J_K(.)\dot{K}\right]A - J(.)\dot{A}}{A^2} = 0 \tag{A.10}$$

$$\eta_A \hat{A} + \eta_L \hat{L} + \eta_K \hat{K} - \hat{A} = 0 \tag{A.11}$$

where $\eta_x := \frac{J_x(.)x}{J(.)}$ for $x = A, K, L$.

Equations (A.9) and (A.11) yield a system of linear equations in $\hat{K}$ and $\hat{A}$, which is restated in a slightly modified form for the readers convenience.

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[28] At this point it is appropriate to differentiate $F(.)$ with respect to the physical stocks of factor inputs. Recall that the allocation variables are constant along the balanced-growth path.

$$(1-\sigma_K)\hat{K}-\sigma_A\hat{A}=\sigma_L n \tag{A.12}$$

$$(1-\eta_A)\hat{A}-\eta_K\hat{K}=\eta_L n \tag{A.13}$$

Provided that $n>0$ this system of equations is inhomogeneous and uniquely determines $\hat{K}$ and $\hat{A}$ in terms of the underlying parameters (σ_X and η_X are constant in the Cobb-Douglas case). The solution is given by $\hat{K}=\dfrac{\left[\sigma_L(1-\eta_A)+\eta_L\sigma_A\right]n}{(1-\eta_A)(1-\sigma_K)-\eta_K\sigma_A}$ and $\hat{A}=\dfrac{\left[\eta_L(1-\sigma_K)+\eta_K\sigma_L\right]n}{(1-\eta_A)(1-\sigma_K)-\eta_K\sigma_A}$.

5.3. A general R&D-based growth model: the social solution

5.3.1. Dynamic problem, first-order conditions and dynamic system

The social solution for the class of models under study is derived using a general formulation (apart from preferences). The social planner's problem may be expressed as follows (see also Eicher and Turnovsky, 1999a).

$$\max_{\{C/L,\,\theta,\,\phi\}}\int_0^\infty \frac{(C/L)^{1-\gamma}-1}{1-\gamma}e^{-\rho t}\,dt \tag{A.14}$$

$$s.t.\;\; Y=F(A,\theta L,\phi K) \tag{A.15}$$

$$\dot{K}=Y-C-\delta_K K \tag{A.16}$$

$$\dot{A}=J\left[A,(1-\theta)L,(1-\phi)K\right] \tag{A.17}$$

$$K(0)>0,\; A(0)>0 \tag{A.18}$$

The current-value Hamiltonian together with the (necessary) optimality conditions are displayed below. The costate variables of capital and technology are denoted by μ_K and μ_A, respectively.

$$H\left(C/L,\,\theta,\,\phi,\,K,\,A,\,\mu_K,\,\mu_A\right):= \frac{\left(C/L\right)^{1-\gamma}-1}{1-\gamma}+\mu_K\left[F\left(A,\,\theta L,\,\phi K\right)-C-\delta_K\,K\right]+\mu_A J\left[A,\left(1-\theta\right)L,\left(1-\phi\right)K\right] \tag{A.19}$$

$$H_{C/L}=\left(C/L\right)^{-\gamma}-\mu_K\,L=0 \quad\Leftrightarrow\quad C^{-\gamma}=\mu_K\,L^{1-\gamma} \qquad \text{(maximum principle 1)} \tag{A.20}$$

$$H_\theta=\mu_K\,F_\theta(.)+\mu_A\,J_\theta(.)=\mu_K\,F_\theta(.)-\mu_A\,J_{1-\theta}(.)=0 \qquad \text{(maximum principle 2)}$$
$$\Leftrightarrow\quad \mu_K\,F_\theta(.)=\mu_A\,J_{1-\theta}(.) \tag{A.21}$$

$$H_\phi=\mu_K\,F_\phi(.)+\mu_A\,J_\phi(.)=\mu_K\,F_\phi(.)-\mu_A\,J_{1-\phi}(.)=0 \qquad \text{(maximum principle 3)}$$
$$\Leftrightarrow\quad \mu_K\,F_\phi(.)=\mu_A\,J_{1-\phi}(.) \tag{A.22}$$

$$\dot{\mu}_K=-H_K+\rho\,\mu_K=-\left[\mu_K\,F_K(.)-\mu_K\,\delta_K+\mu_A\,J_K(.)\right]+\rho\,\mu_K \qquad \text{(costate equation 1)}$$
$$\Leftrightarrow\quad \frac{\dot{\mu}_K}{\mu_K}=\rho+\delta_K-F_K(.)-\frac{\mu_A}{\mu_K}J_K(.) \tag{A.23}$$

$$\dot{\mu}_A=-H_A+\rho\,\mu_A=-\left[\mu_K\,F_A(.)+\mu_A\,J_A(.)\right]+\rho\,\mu_A \qquad \text{(costate equation 2)}$$
$$\Leftrightarrow\quad \frac{\dot{\mu}_A}{\mu_A}=\rho-J_A(.)-\frac{\mu_K}{\mu_A}F_A(.) \tag{A.24}$$

$$\dot{K}=H_{\mu_K}=F(.)-C-\delta_K\,K \qquad \text{(state equation 1)} \tag{A.25}$$

$$\dot{A}=H_{\mu_A}=J(.) \qquad \text{(state equation 2)} \tag{A.26}$$

$$\lim_{t\to\infty}e^{-\rho t}K\,\mu_K=0\,,\ \lim_{t\to\infty}e^{-\rho t}A\,\mu_A=0 \qquad \text{(transversality conditions)} \tag{A.27}$$

With respect to “maximum principle 2 and 3” it should be noted that $F_\theta(.)=-F_{1-\theta}(.)$ and $J_\theta(.)=-J_{1-\theta}(.)$. Moreover, the formulation of the effective amount of factor inputs, ϕK for example, requires to differentiate the Hamiltonian with respect to the control variable, ϕ, to apply the respective maximum principle and with respect to the state variable, K, to derive the respective costate equation (see also Eicher and Turnovsky, 1999b, p. 424). In evaluating these derivatives it should be remembered that $F_\phi\left(A,\theta L,\phi K\right)=F_{\phi K}(.)\,K$ and similarly $F_K\left(A,\theta L,\phi K\right)=F_{\phi K}(.)\,\phi$. From (A.20) together with (A.23) one can easily derive the differential equation in C: $\dot{C} = \frac{C}{\gamma}\left[F_K(.) - \delta_K - \rho - (1-\gamma)\,n + \frac{\mu_A}{\mu_K} J_K(.) \right]$; the last term in brackets, $\frac{\mu_A}{\mu_K} J_K(.)$, is the marginal product of capital in R&D multiplied by ϕ [i.e. $J_K(.) = J_{\phi K}(.)\,\phi$], in units of the final output good. The dynamic system can be summarised as follows.

$$\dot{K} = F(A,\, \theta L,\, \phi K) - \delta_K\, K - C \tag{A.28}$$

$$\dot{A} = J\left[A,\, (1-\theta) L,\, (1-\phi) K \right] \tag{A.29}$$

$$\dot{C} = \frac{C}{\gamma}\left[F_K(.) - \delta_K - \rho - (1-\gamma)\,n + \frac{\mu_A}{\mu_K} J_K(.) \right] \tag{A.30}$$

$$\frac{\dot{\mu}_K}{\mu_K} = \rho + \delta_K - F_K(.) - \frac{\mu_A}{\mu_K} J_K(.) \tag{A.31}$$

$$\frac{\dot{\mu}_A}{\mu_A} = \rho - J_A(.) - \frac{\mu_K}{\mu_A} F_A(.) \tag{A.32}$$

$$\mu_K\, F_\theta(.) = \mu_A\, J_{1-\theta}(.) \tag{A.33}$$

$$\mu_K\, F_\phi(.) = \mu_A\, J_{1-\phi}(.) \tag{A.34}$$

5.3.2. Dynamic system in scale-adjusted variables

Provided that $n > 0$ the balanced growth rates are given by $\hat{Y} = \hat{K} = \hat{C} = \beta_K\, n$ and $\hat{A} = \hat{J} = \beta_A\, n$. The appropriate scale adjustments read as follows $y := Y / L^{\beta_K}$, $k := K / L^{\beta_K}$, $c := C / L^{\beta_K}$, $a := A / L^{\beta_A}$ and $j := J / L^{\beta_A}$. Furthermore, from (A.32) it follows that $\frac{\dot{\mu}_A}{\mu_A} = \rho - \eta_A \frac{J}{A} - \frac{\mu_K}{\mu_A}\, \sigma_A \frac{Y}{A}$. Along a balanced growth path, $\hat{\mu}_A$ must be constant. The third term on the RHS is a linear transform of $\hat{A}$ and, hence, constant along a balanced growth path. Accordingly, the last term on the RHS must be constant either, implying that $\hat{\mu}_K - \hat{\mu}_A = n(\beta_A - \beta_K) = \hat{A} - \hat{K}$. We can reduce the order of the system under study by taking the ratio of the two costate variables. The appropriate scale-adjustment for this ratio is given by $s := \frac{\mu_K}{\mu_A\, L^{\beta_A - \beta_K}}$. Differentiating this definition with respect to time yields

$\hat{s} = \hat{\mu}_K - \hat{\mu}_A - n\left(\beta_A - \beta_K\right)$. The next step is derive expressions for $\hat{\mu}_K$ and $\hat{\mu}_A$ in terms of scale-adjusted variables and insert these into the preceding equation. Taking the efficiency condition $s\frac{\sigma_K\, y}{\phi} = \frac{\eta_K\, j}{1-\phi}$ into account and noting that $\frac{\mu_A}{\mu_K} = L^{\beta_K - \beta_A} / s$, $\hat{\mu}_K$ can be expressed to read $\hat{\mu}_K = \rho + \delta_K - \frac{\sigma_K y}{k}\left(1 - \frac{1-\phi}{\phi}\right)$. Similarly, by noting that $s\frac{\sigma_L\, y}{\theta} = \frac{\eta_L\, j}{1-\theta}$ and $\frac{\mu_K}{\mu_A} = s\, L^{\beta_A - \beta_K}$, $\hat{\mu}_A$ can be written as $\hat{\mu}_A = \rho - \frac{j}{a}\left(\eta_A + \frac{\sigma_A\, \eta_L}{\sigma_L}\frac{\theta}{1-\theta}\right)$. The system in scale-adjusted variables may then be expressed as follows.

$$\dot{k} = y - c - \delta_K\, k - \beta_K\, n\, k \tag{A.35}$$

$$\dot{a} = j - \beta_A\, n\, a \tag{A.36}$$

$$\dot{c} = \frac{c}{\gamma}\left[\frac{\sigma_K\, y}{k} - \delta_K - \rho - (1-\gamma)\, n + \frac{\eta_K\, j}{s\, k}\right] - \beta_K\, n\, c \tag{A.37}$$

$$\dot{s} = s\left[\frac{j}{a}\left(\eta_A + \frac{\sigma_A\, \eta_L}{\sigma_L}\frac{\theta}{1-\theta}\right) - \frac{\sigma_K\, y}{k}\left(1 - \frac{1-\phi}{\phi}\right) - n\,(\beta_A - \beta_K) + \delta_K\right] \tag{A.38}$$

$$s\frac{\sigma_L\, y}{\theta} = \frac{\eta_L\, j}{1-\theta} \tag{A.39}$$

$$s\frac{\sigma_K\, y}{\phi} = \frac{\eta_K\, j}{1-\phi} \tag{A.40}$$

5.3.3. The basic non-scale model

Using the output functions in scale-adjusted variables [$y = \alpha_F a^{\sigma_A} \theta^{\sigma_L} k^{\sigma_K}$ and $j = \alpha_J a^{\eta_A} (1-\theta)^{\eta_L}$] together with the general system shown in 6.3.2., the dynamic system in scale-adjusted variables for the basic non-scale model turns out to read as follows (compare to Eicher and Turnovsky, 1999b, p. 424).

$$\dot{k} = \alpha_F\, a^{\sigma_A}\, \theta^{\sigma_L}\, k^{\sigma_K} - c - \delta_K\, k - \beta_K\, n\, k \tag{A.41}$$

$$\dot{a} = \alpha_J\, a^{\eta_A}\, (1-\theta)^{\eta_L} - \beta_A\, n\, a \tag{A.42}$$

$$\dot{c} = \frac{c}{\gamma}\Big[\sigma_K\, \alpha_F\, a^{\sigma_A}\, \theta^{\sigma_L}\, k^{\sigma_K - 1} - \delta_K - \rho - (1-\gamma)\, n\ \Big] - \beta_K\, n\, c \tag{A.43}$$

$$\dot{s} = s\left[\ \alpha_J\, a^{\eta_A - 1}\, (1-\theta)^{\eta_L} \left(\eta_A + \frac{\sigma_A\, \eta_L}{\sigma_L}\frac{\theta}{1-\theta}\right) - \sigma_K\, \alpha_F\, a^{\sigma_A}\, \theta^{\sigma_L}\, k^{\sigma_K - 1} - n\,(\beta_A - \beta_K) + \delta_K\ \right] \tag{A.44}$$

$$s\, \sigma_L\, \alpha_F\, a^{\sigma_A}\, \theta^{\sigma_L - 1}\, k^{\sigma_K} = \eta_L\, \alpha_J\, a^{\eta_A}\, (1-\theta)^{\eta_L - 1} \tag{A.45}$$

with $\beta_K = \dfrac{(1-\eta_A)\sigma_L + \eta_L \sigma_A}{(1-\eta_A)(1-\sigma_K)}$ and $\beta_A = \dfrac{\eta_L}{1-\eta_A}$.

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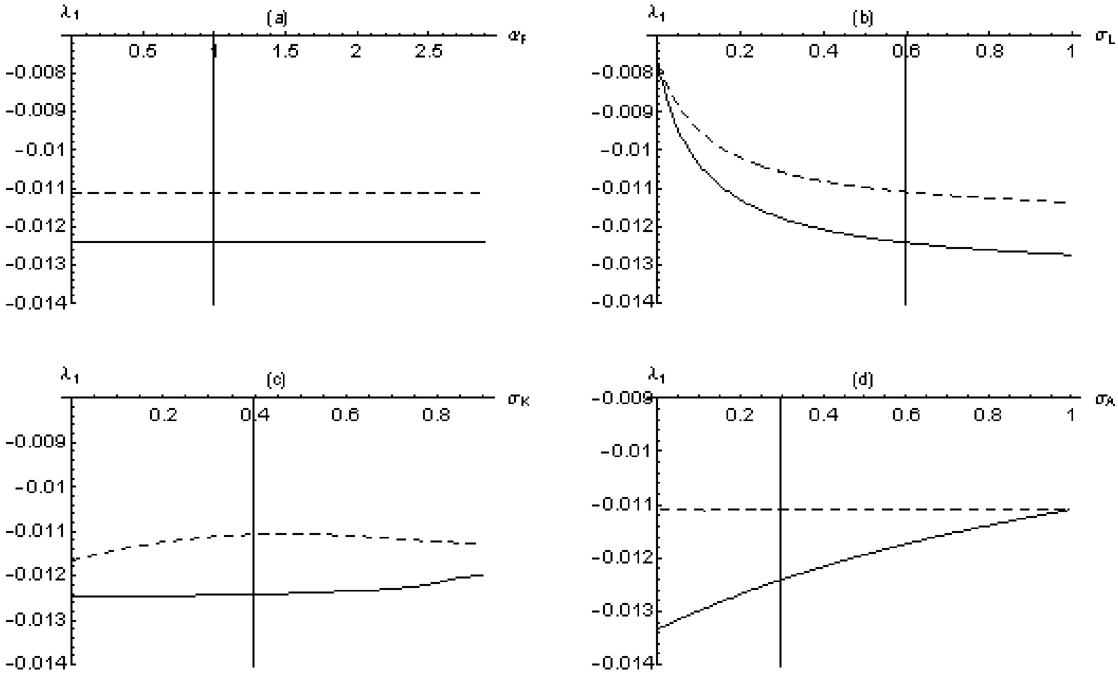

Figure 1: Dominant eigenvalue for the social (solid) and decentral (dashed) solution in response to variations in final output technology parameters.

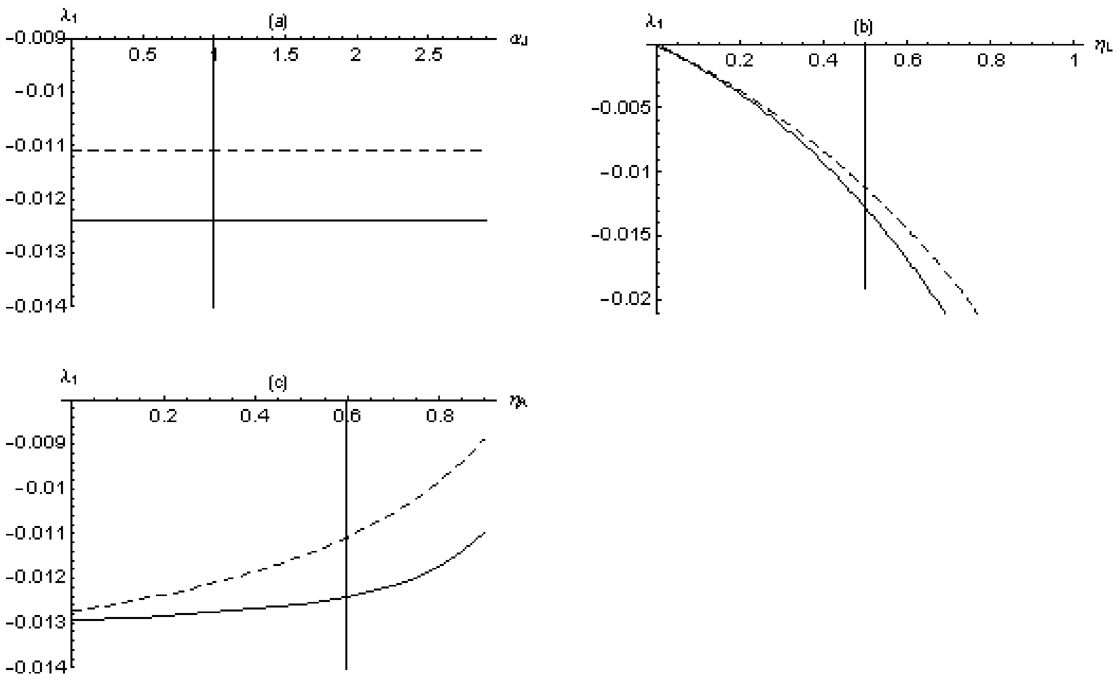

Figure 2: Dominant eigenvalue for the social (solid) and decentral (dashed) solution in response to variations in R&D technology parameters.

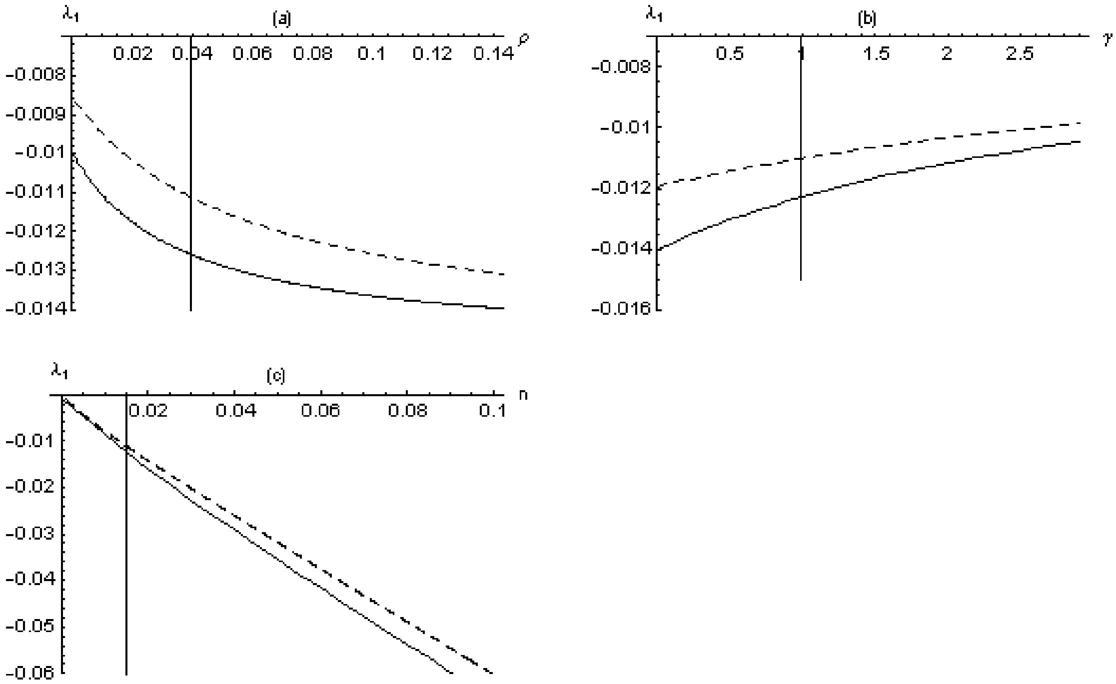

Figure 3: Dominant eigenvalue for the social (solid) and decentral (dashed) solution in response to variations in preference parameters and the population growth rate.

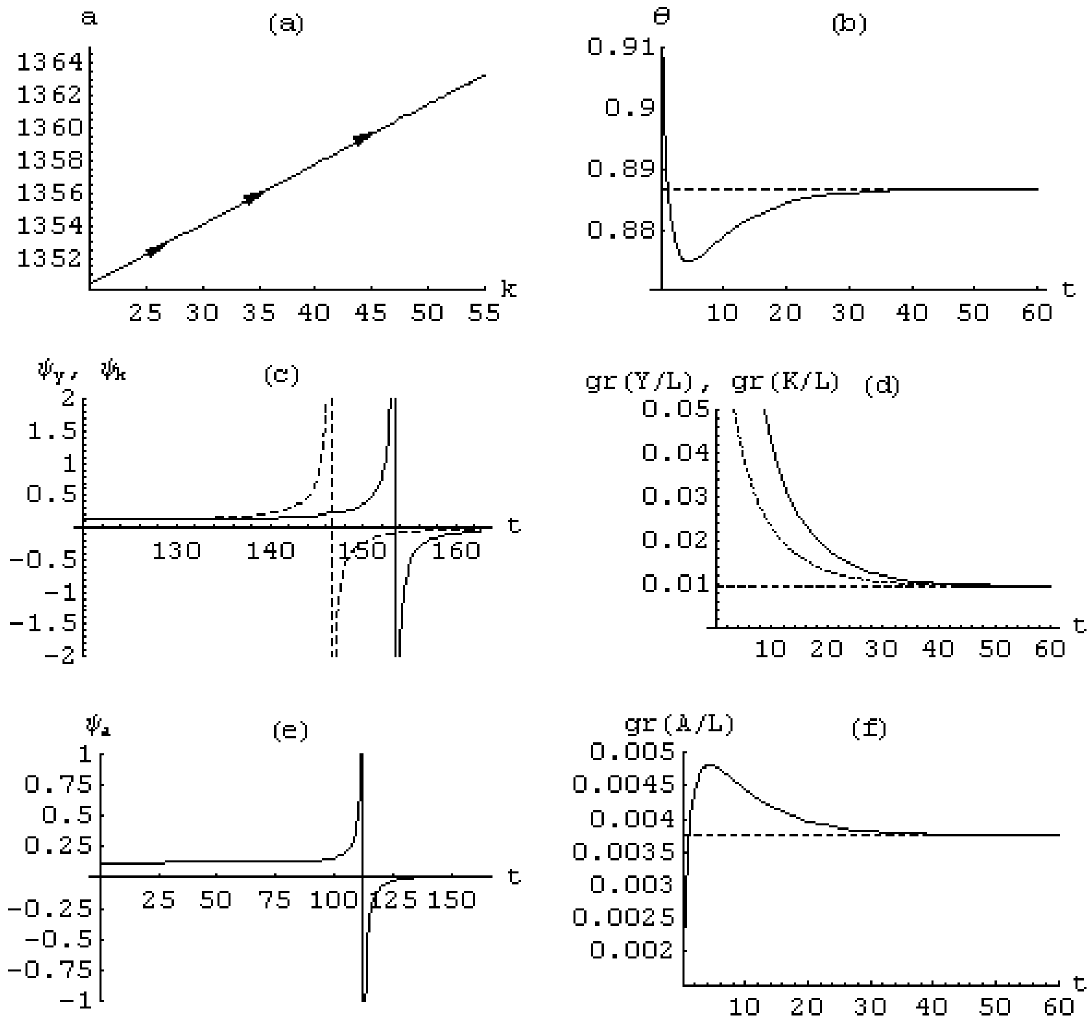

Figure 4: Illustration of transitional dynamics (adjustment from below, basic model).

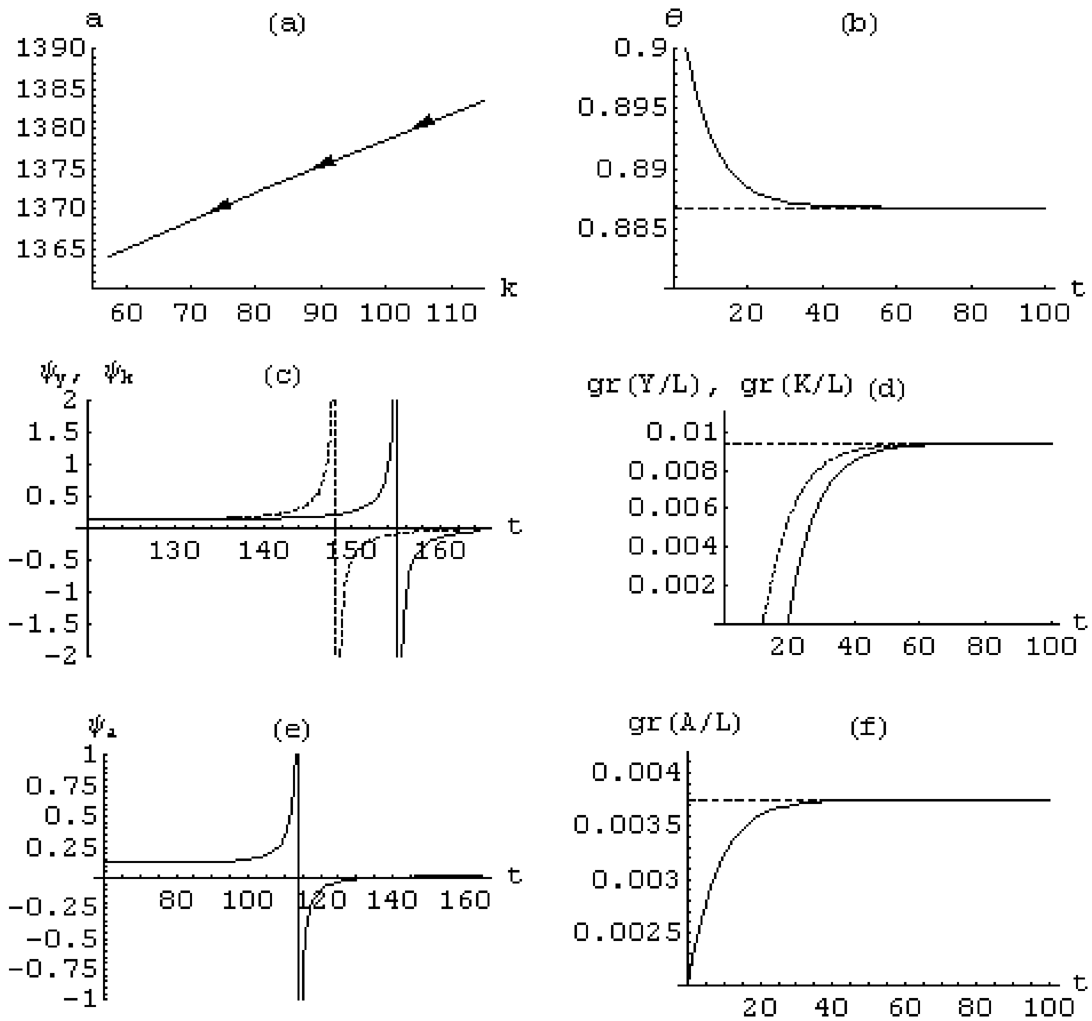

Figure 5: Illustration of transitional dynamics (adjustment from above, basic model).

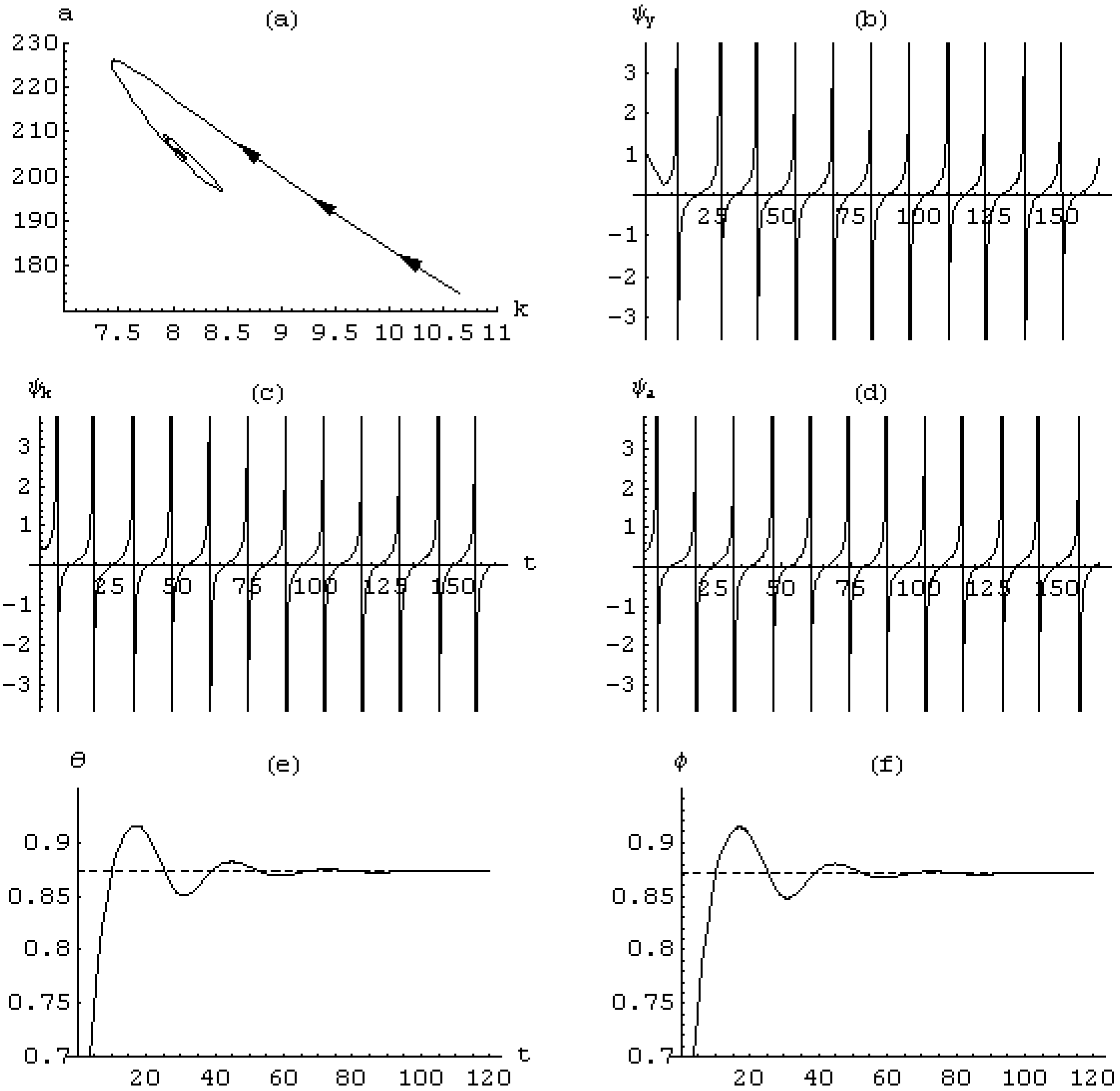

Figure 6: Illustration of transitional dynamics (generalised R&D-based model).

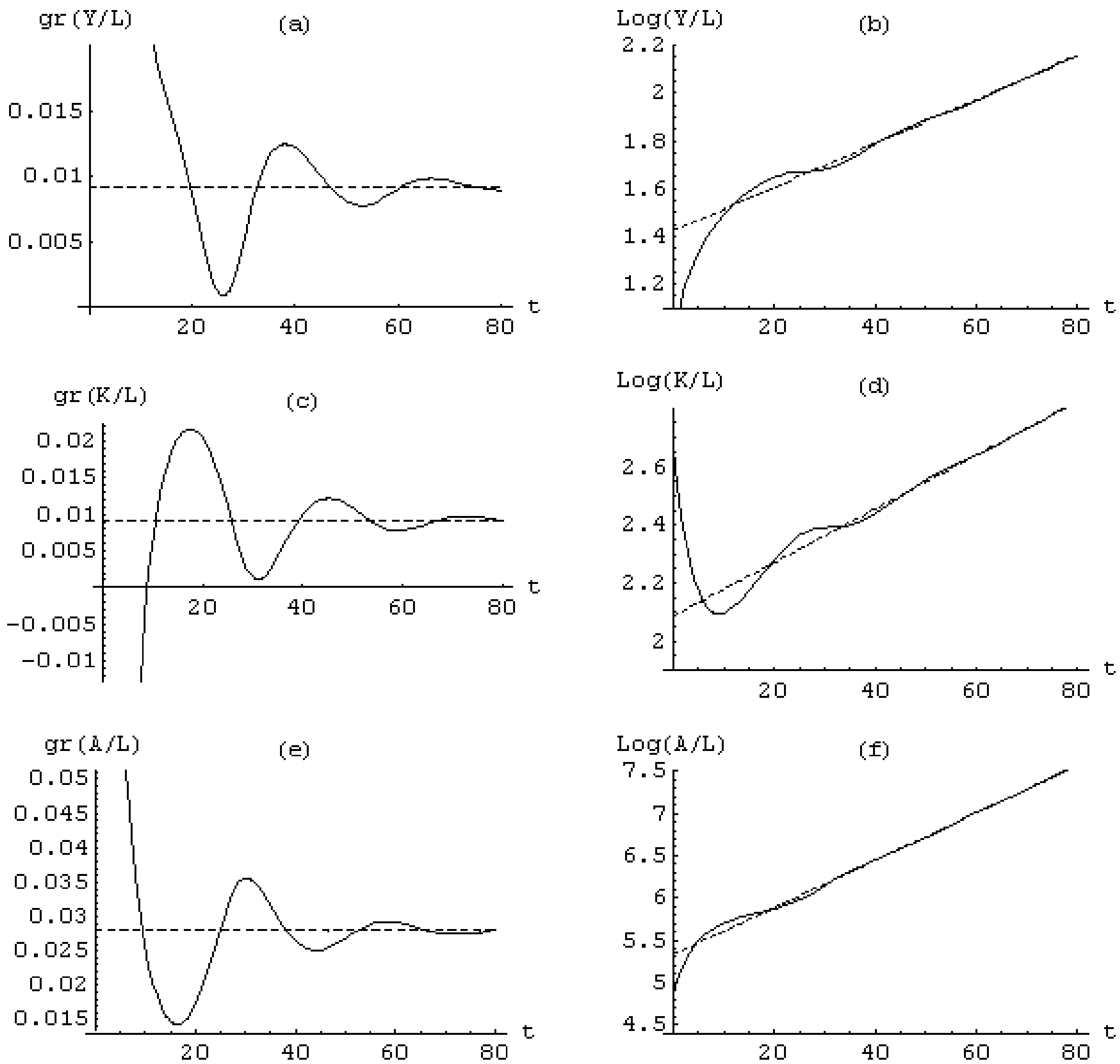

Figure 7: Transitional dynamics in observable variables (generalised R&D-based model).

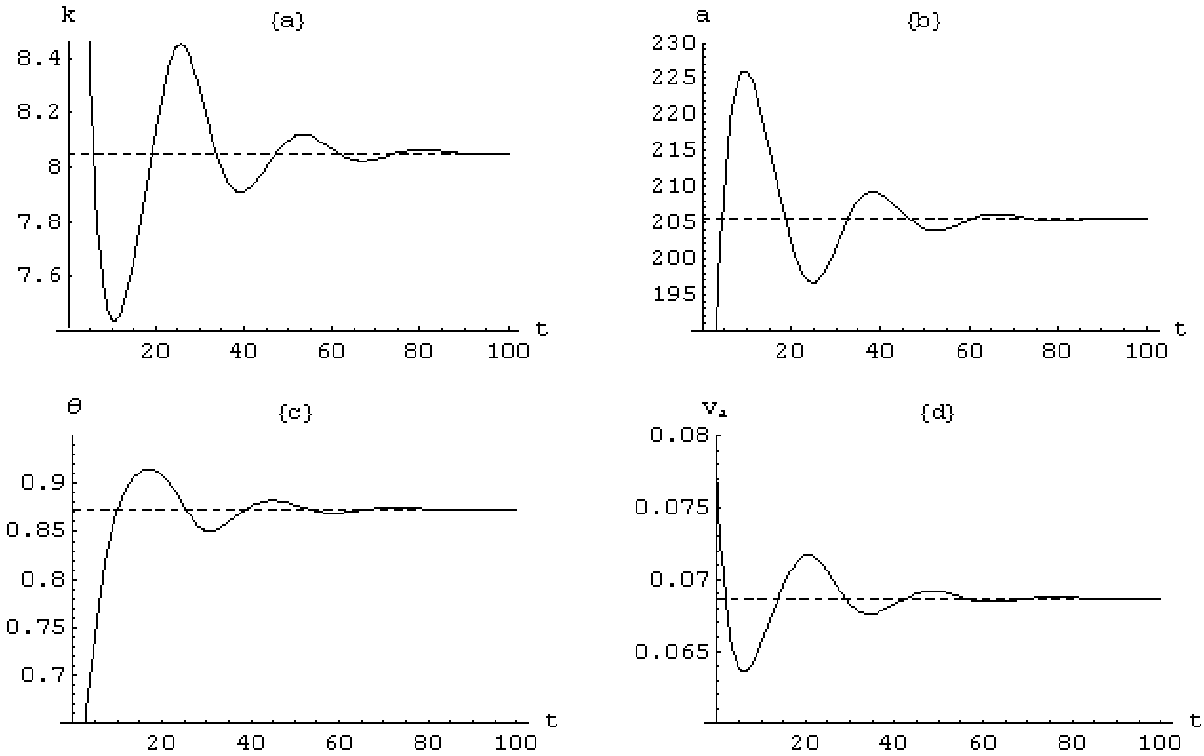

Figure 8: The economic intuition behind the non-monotonic adjustment (generalised R&D-based model).

Table 1: Baseline set of parameters (basic non-scale model).

FO technology (IO technology):*	$\alpha_F = 1$; $\sigma_L = 0.6$; $\sigma_K = 0.4$; $\sigma_A = 0.3$; $\delta_K = 0.05$; $q = 1$
R&D technology:	$\alpha_J = 1$; $\eta_L = 0.5$ ($\eta_L^p = 1$, $\eta_L^e = -0.5$); $\eta_A = 0.6$
Preferences and population growth:	$\rho = 0.04$; $\gamma = 1$; $n = 0.015$

*FO: final output; IO: intermediate output

Table 2: Growth rates, key economic ratios and rate of convergence (basic non-scale model).

	$\widehat{Y/L}$	$\widehat{A/L}$	$\widetilde{Y/K}$	$\widetilde{C/Y}$	$\tilde{y}$ *	$\tilde{\theta}$	$\tilde{\psi}_{Y/L}$
Market solution:	0.009	0.004	0.71	0.89	41	0.89	0.0050
Social solution:	0.009	0.004	0.29	0.74	67	0.91	0.0067

*: expressed in units of final output

Table 3: Baseline set of parameters (generalised non-scale model).

FO technology (IO technology):*	$\alpha_F = 1$; $\sigma_L = 0.56$; $\sigma_K = 0.44$; $\sigma_A = 0.12$; $\delta_K = 0.05$; $q = 1$
R&D technology:	$\alpha_J = 1$; $\eta_L = 0.5$ ($\eta_L^p = 0.56$, $\eta_L^e = -0.06$); $\eta_K = 0.4$ ($\eta_K^p = 0.44$, $\eta_K^e = -0.04$); $\eta_A = 0.6$
Preferences and population growth:	$\rho = 0.04$; $\gamma = 1$; $n = 0.015$

*FO: final output; IO: intermediate output

Table 4: Growth rates, key economic ratios and rate of convergence (generalised non-scale model, market solution).

$\widehat{Y/L}$	$\widehat{A/L}$	$\widetilde{Y/K}$	$\widetilde{C/Y}$	$\tilde{\theta}$	$\tilde{\phi}$	$\tilde{\psi}_{Y/L}$
0.009	0.028	0.51	0.85	0.87	0.87	0.027